Regenerative compositions in the case of slow variation

Barbour, A D ; Gnedin, A

Abstract: For S a subordinator and Πn an independent Poisson process of intensity ne ,x>0 we are interested in the number Kn of gaps in the range of S that are hit by at least one point of Πn. Extending previous studies in [A.V. Gnedin, The Bernoulli sieve, Bernoulli 10 (2004) 79–96; A.V. Gnedin, J. Pitman, M. Yor, Asymptotic laws for compositions derived from transformed subordinators, Ann. Probab. 2006 (in press). http://arxiv.org/abs/math.PR/0403438, 2004; A.V. Gnedin, J. Pitman, M. Yor, Asymptotic laws for regenerative compositions: gamma subordinators and the like, Probab. Theory Related Fields (2006)] we focus on the case when the tail of the Lévy measure of S is slowly varying. We view Kn as the terminal value of a random process , and provide an asymptotic analysis of the fluctuations of , Kn, as n→∞, for a wide spectrum of situations.

DOI: https://doi.org/10.1016/j.spa.2005.12.006

Originally published at:
Barbour, A D; Gnedin, A (2006). Regenerative compositions in the case of slow variation. Stochastic Processes and their Applications, 116(7):1012-1047.
DOI: https://doi.org/10.1016/j.spa.2005.12.006

Regenerative Compositions in the Case of Slow Variation

A.D. Barbour* and A.V. Gnedin†
Universität Zürich and Universiteit Utrecht

Abstract

For S a subordinator and Π_n an independent Poisson process of intensity $ne^{-x}, x > 0$, we are interested in the number K_n of gaps in the range of S that are hit by at least one point of Π_n. Extending previous studies in [7, 10, 11] we focus on the case when the tail of the Lévy measure of S is slowly varying. We view K_n as the terminal value of a random process $\mathcal{K}_n$, and provide an asymptotic analysis of the fluctuations of $\mathcal{K}_n$, as $n \to \infty$, for a wide spectrum of situations.

1 Introduction

Let $S = (S_t, t \geq 0)$ be an increasing Lévy process (subordinator) with $S_0 = 0$, zero drift and no killing. The closed range $\mathcal{R}$ of S has zero Lebesgue measure, and defines a random division of the complement set $\mathbb{R}_+ \setminus \mathcal{R}$ into open interval components, referred to as *gaps*. In this paper, we are concerned with the distribution of the number K_n of gaps hit by at least one point of an independent Poisson process Π_n with the inhomogeneous rate ne^{-x}, $x > 0$, where n is a large parameter. We actually go into more detail. We view K_n as the terminal value $\mathcal{K}_n(\infty)$ of the increasing process $\mathcal{K}_n = (\mathcal{K}_n(T),\, T \geq 0)$, where $\mathcal{K}_n(T)$ is defined to be the number of jumps of the subordinator within $[0, T]$ which cover one or more Poisson points; that is, $\mathcal{K}_n(T)$ counts all instants $t \in [0, T]$ which satisfy $\Pi_n \cap]S_{t-}, S_t[\neq \emptyset$. Our aim is to describe the random fluctuations of the process $\mathcal{K}_n$.

The general motivation for the setting stems from the study of the number of blocks in a random decomposable combinatorial structure, which in the case under focus is a *composition* (ordered partition) $\mathcal{C}_n$ of some integer. Viewing $\mathcal{C}_n$ as distribution of some number of balls in some collection of boxes, each gap may be interpreted as a box, which is hit by a particular ball with probability equal to the exponential measure of the gap. The parameter n controls the total number of balls, which is a Poisson variable, and the composition $\mathcal{C}_n$ of this random number is defined as the consecutive record of nonzero occupancy numbers, in the natural ordering of the gaps. Our study fits into the recent

*Angewandte Mathematik, Winterthurerstrasse 190, CH–8057 Zürich, Switzerland; supported in part by Schweizer Nationalfonds Projekt Nr. 20-107935/1

†Mathematisch Instituut, PO Box 80010, 3508 TA Utrecht, Nederland

theory of sampling models called *regenerative composition structures*, which have a distinguished Markovian property resulting from the renewal features of $\mathcal{R}$ combined with that of the exponential distribution [8, 9]. Concretely, the regeneration property of $\mathcal{C}_n$ means that, for each $t > 0$, conditionally given the value $s = S_t$, the partial compositions appearing within $[0, s]$ and $[s, \infty[$ are independent and the latter has the same distribution as the composition $\mathcal{C}_{ne^{-s}}$.

The distribution of S is completely determined by a Lévy measure ν_0 on $\mathbb{R}_+$, which describes the intensity of the jumps of different sizes, and the behaviour of $\mathcal{K}_n$ depends very much on the form of ν_0. Qualitatively different modes of behaviour are known [7, 10, 11].

For ν_0 a finite measure, S is a compound Poisson process. Under mild additional assumptions, the two central moments are of the order of $\log n$ and K_n is asymptotically normal. In this situation, the methods of renewal theory are adequate, since the process $\mathcal{K}_n(T)$ essentially coincides with the process of jump epochs of S for $T < \log n$, while the contribution of larger times $T > \log n$ to K_n is negligible, see [7]. In particular, when ν_0 is an exponential distribution, the induced composition follows the poissonised (ordered) Ewens sampling formula, in which case much finer results on K_n are available by combinatorial methods [1, 14].

If ν_0 is infinite, and its tail $N_0(x) := \nu_0[x, \infty[$ is such that $N(1/y)$ is regularly varying as $y \to \infty$ with exponent α (here and henceforth this means regular variation with $0 < \alpha \leq 1$), then $\mathbb{E}K_n$ is also regularly varying with the same exponent and $K_n/\mathbb{E}K_n$ approaches a nondegenerate limit, which is not gaussian. The moments of $\mathcal{K}_n(T)$ are then of the same order of magnitude as that of K_n, for each fixed T, see [10].

Between these two possibilities lies the setting in which $N_0(1/y)$ is slowly varying as $y \to \infty$, but the Lévy measure is infinite, *i.e.* $\lim_{y\to\infty} N_0(1/y) = \infty$. Here, the special case with $N_0(1/y) \sim c \log y$ has been studied in some detail. For these *gamma-like* subordinators, the proper formats for the two central moments of K_n are $\log^2 n$ and $\log^3 n$, respectively, and the limiting distribution is again normal, see [11].

In this paper, we treat the case of slowly varying N_0 in greater generality. As might be expected of a transitional régime between the finite and the regularly varying cases, there is a further wealth of possible modes of behaviour, and the discussion reveals how these are related to the time scales over which the significant variation in $\mathcal{K}_n$ occurs. Our argument leading to a functional central limit theorem is very different from that in [7, 11], and is based on the observation that, to first order, the fluctuations of the counting process $\mathcal{K}_n$ are dominated by those of its compensator A_n, defined in Proposition 2.1. The explicit representation of the random process A_n makes it possible to find approximations by rather direct arguments, and under relatively mild conditions. These are broadly speaking of two kinds. The first is expressed in Assumption A2, which puts a mild restriction on the way in which a certain transform L of the measure ν_0 can vary locally as a function of its parameter. Conditions of the second kind, appearing in different forms in (4.36), (4.46), (4.47) and (5.52), limit the global variability of L.

Our analysis of subordinators with slowly varying N_0 distinguishes three basic modes. In the case of *moderate growth*, which includes the subordinators with logarithmic asymptotics $N_0(1/y) \asymp (\log y)^\beta$, $\beta > 0$, including the gamma-like subordinators studied in [11], the random fluctuations of $\mathcal{K}_n(T)$ occur more or less evenly on the scale $T = v \log n$ in

$v \in [0,1]$. In the case of *fast growth*, well exemplified by $N_0(1/y) \asymp \exp(\log^\beta y)$, $0 < \beta < 1$, almost everything happens at times of order $L(n)$, and $L(n)$ is of smaller order than $\log n$. The third case is that of *slow growth*, as for example $N_0(1/y) \asymp \log\log y$, when significant contributions to the random fluctuations of K_n are only made at times very close to $\log n$, just as in the compound Poisson case [7].

Notation. We use λ, λ_j for positive constants whose value is not important and may depend on the context. The asymptotic relation $a_n \asymp b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$, while $a_n \gg b_n$ means that $b_n = o(a_n)$. Asymptotic relations like $X_n \sim Y_n$ or $X_n \asymp Y_n$ for random quantities mean that they hold with probability one, unless otherwise specified.

2 The basic setting

2.1 Laplace exponents and the compensator

The Lévy measure ν_0 is uniqely determined by the Laplace exponent Φ_0, defined for $m \geq 0$ by

$$\Phi_0(m) \;=\; \int_0^\infty (1 - e^{-mx})\, \nu_0(\mathrm{d}x) \;=\; m \int_0^\infty e^{-mx} N_0(x)\, \mathrm{d}x\,;$$

note that ν_0 must satisfy $\Phi_0(1) < \infty$. The distribution of the subordinator is determined by the Lévy-Khintchine formula for the Laplace transform

$$\mathbb{E}\, e^{-nS_t} = e^{-t\Phi_0(n)}\,, \qquad n \geq 0, \tag{2.1}$$

see [2] as a general reference on the Lévy processes and see [3] especially for subordinators.

The function Φ_0 can be extended to an analytic function in the right half-plane, and hence, by Müntz's theorem, Φ_0 can be uniquely extrapolated from the values $\Phi_0(m)$, $m = 1, 2, \ldots$; these also determine the poissonised version of Φ_0, defined either by the series

$$\Phi(n) \;:=\; e^{-n} \sum_{m=1}^\infty \frac{n^m}{m!} \Phi_0(m)\,, \tag{2.2}$$

or by the integral

$$\Phi(n) \;=\; \int_0^\infty (1 - e^{-n(1-e^{-x})})\, \nu_0(\mathrm{d}x). \tag{2.3}$$

This latter transform is particularly useful to us, since it appears naturally in the definition of the compensator A_n of the counting process $\mathcal{K}_n$.

Proposition 2.1 *With respect to the filtration $(\mathcal{F}_{T,n},\, T \geq 0)$, defined by*

$$\mathcal{F}_{T,n} \;:=\; \sigma\left\{S_t,\, 0 \leq t \leq T;\; \Pi_n|_{[0,S_T]}\right\},$$

the compensator of $\mathcal{K}_n$ is the increasing process A_n given by the formula

$$A_n(T) \;:=\; \int_0^T \Phi(ne^{-S_t})\, \mathrm{d}t\,, \qquad T \in [0,\infty]\,. \tag{2.4}$$

Proof. The subordinator gains an increment within $[x, x + \mathrm{d}x]$ at rate $\nu_0(\mathrm{d}x)$. On the other hand, Π_n hits $[S_{t-}, S_{t-} + x]$ with probability $1 - \exp(-ne^{-s}(1 - e^{-x}))$ (where $s = S_{t-}$), because the number of atoms in $[s\,, s + x]$ has Poisson distribution with mean

$$n \int_s^{s+x} e^{-u}\, \mathrm{d}u = ne^{-s}(1 - e^{-x})\,.$$

Integrating over x yields the derivative $\mathrm{d}A_n(t)/\mathrm{d}t = \Phi(ne^{-S_t})$. □

We further assume that

$$\mathbb{E}S_t = t \quad \text{and} \quad \mathrm{Var}\, S_t = t\sigma^2\,.$$

The former is the same as

$$\mathbb{E}S_1 = \Phi_0'(0) = \int_0^\infty x\, \nu_0(\mathrm{d}x) = \int_0^\infty N_0(x)\, \mathrm{d}x = 1\,. \tag{2.5}$$

This can always be achieved by a linear time-scaling, which does not affect the range $\mathcal{R}$. A consequence of the assumption is that $N_0(x)$ is a probability density. It then follows for all $m \geq 0$ that

$$\Phi_0(m) \;\leq\; m \int_0^\infty N_0(x)\, \mathrm{d}x \;=\; m \quad \text{and} \quad \Phi(m) \;\leq\; m, \tag{2.6}$$

this last from (2.2).

With this scaling, $N_0(x)$ is the density of a *delay* variable. If X has this density and is independent of S, then the process $(X + S_t,\, t \geq 0)$ is a *stationary* subordinator, in the sense that its closed range $X + \mathcal{R}$ may be extended to a random subset of $\mathbb{R}$ invariant under all translations. In particular, $X + S_t$ has the same overshoot distribution at every level $s \geq 0$.

It will be convenient to define all the Poisson processes $(\Pi_n,\, n \geq 0)$ (which are independent of the subordinator S) consistently on the same probability space. To this end, we take an inhomogeneous planar Poisson point process on $\mathbb{R}_+^2$ with intensity measure $e^{-y}\, \mathrm{d}y\, \mathrm{d}n$, and we introduce Π_n as the projection on the y-axis of the planar process restricted to the strip $[0, \infty] \times [0, n]$. In this setting, the compositions $\mathcal{C}_n$ are defined consistently for all $n \geq 0$: a decrease in n has the effect of thinning, *i.e.* removing some balls from the boxes; while as n increases more Poisson atoms are added, hence $\mathcal{K}_n(T)$ and K_n are nondecreasing in n. Thus, in principle, our setting is 3-dimensional, with three parameters n, t, s meaning the intensity, the time and the range of subordinator.

For our analysis of A_n, it is also convenient to note that we can truncate the integral (2.4), which defines $A_n(T)$, at the first passage time

$$\tau_n \;:=\; \min\{t \colon S_t \geq \log n\}, \tag{2.7}$$

with little loss.

Lemma 2.2 *The jumps of S after τ_n make only a bounded contribution to $\mathcal{K}_n(T)$, uniformly in $T \leq \infty$, and for $\psi > 0$*

$$\mathbb{P}\left[\sup_{T \geq 0} |A_n(T) - A_n(T \wedge \tau_n)| > \psi\right] \;\leq\; \psi^{-1} \int_0^\infty \mathbb{E}\left(e^{-S_t}\right)\, \mathrm{d}t.$$

Proof. $K_n - \mathcal{K}_n(\tau_n)$ cannot exceed the number of atoms of $\Pi_n \cap [\log n, \infty[$, which is Poisson distributed with mean 1. To estimate the contribution to the compensator, recalling (2.6), we have

$$\int_{\tau_n}^{\infty} \Phi(ne^{-S_t})\,\mathrm{d}t \;<\; n\int_{\tau_n}^{\infty} e^{-S_t}\,\mathrm{d}t \;=\; ne^{-S_{\tau_n}}\int_0^{\infty} e^{-S'_u}\,\mathrm{d}u \;\leq\; \int_0^{\infty} e^{-S'_u}\,\mathrm{d}u\,,$$

where S' defined by $S'_u := S_{\tau_n+u} - S_{\tau_n}$, $u \geq 0$, has the same distribution as S. Markov's inequality completes the proof. $\square$

2.2 Slow variation

Our aim is to investigate the process $\mathcal{K}_n$ in the intermediate setting, between that in which the tail N_0 of ν_0 is regularly varying (with exponent $0 < \alpha \leq 1$), and that in which it is bounded. We therefore assume that

$$\textbf{Assumption A1}: N_0(1/y) \text{ is an unbounded function of slow variation for } y \to \infty. \tag{2.8}$$

The condition can be equally stated in terms of Φ_0, because by the Abel–Tauber theorem [6]:

$$\Phi_0(m) \sim N_0(1/m)\,, \qquad \text{as } \; m \to \infty\,.$$

In what follows, we prefer to work in terms of Φ, because of (2.4), so that A1 is then more naturally expressed in the equivalent form: $\Phi(m)$ is unbounded and slowly varying at infinity. While this equivalence is more or less clear from (2.2), it is useful for Section 3 to have a better idea of how close the functions Φ and Φ_0 are to each other. To this end, we define a measure ν on $[0,1]$ as the pushforward of ν_0 under the change of variable $x \to 1 - e^{-x}$. Then we have

$$\Phi_0(m) = \int_0^1 (1-(1-x)^m)\,\nu(\mathrm{d}x) = m\int_0^1 (1-x)^{m-1}\,N(x)\,\mathrm{d}x\,,$$

where $N(x) := \nu[x,1]$, and

$$\Phi(m) = \int_0^1 (1-e^{-mx})\,\nu(\mathrm{d}x) = m\int_0^1 e^{-mx}\,N(x)\,\mathrm{d}x\,.$$

So the substitution transforms a Laplace integral into a Mellin integral, while Φ assumes the conventional form of a Laplace exponent (hence Φ also corresponds to some subordinator, whose jump-sizes do not exceed 1). We can now use these representations to prove the following lemma.

Lemma 2.3 *Under Assumption* A1, *we have*

$$|\Phi_0^{(l)}(m) - \Phi^{(l)}(m)| = o\left(\frac{\Phi(m)}{m^{l+1}}\right), \quad l \geq 0,$$

where $f^{(l)}$ denotes the lth derivative of f.

Proof. Because $N_0(1/y)$ is slowly varying as $y \to \infty$, the same is true of $N(1/y)$, and $N(1/m) \sim \Phi(m)$. Hence we immediately have

$$\Phi^{(l)}(m) \;=\; O\left(\frac{\Phi(m)}{m^l}\right), \quad l \geq 0. \tag{2.9}$$

We now define

$$D_0(m) \;:=\; m^{-1}\{\Phi(m) - \Phi_0(m)\} = \int_0^1 \{e^{-mx} - e^{(m-1)\log(1-x)}\}\, N(x)\, \mathrm{d}x.$$

Since

$$\begin{aligned}\int_0^1 \{e^{-mx} - e^{m\log(1-x)}\}\, N(x)\, \mathrm{d}x \;&=\; \int_0^1 e^{-mx}(1 - \exp\{m[\log(1-x) + x]\})\, N(x)\, \mathrm{d}x \\ &\sim\; \int_0^1 \tfrac{1}{2} m x^2 e^{-mx} N(x)\, \mathrm{d}x \;\sim\; m^{-2}\Phi(m),\end{aligned}$$

and

$$\int_0^1 e^{m\log(1-x)}\{1 - (1-x)^{-1}\}\, N(x)\, \mathrm{d}x \;\sim\; -\int_0^1 x e^{-mx} N(x)\, \mathrm{d}x \;\sim\; -m^{-2}\Phi(m),$$

it then follows that $|D_0(m)| = o(m^{-2}\Phi(m))$.

For the lth derivative D_l of D_0, we similarly have

$$D_l(m) = \int_0^1 \{(-x)^l e^{-mx} - \{\log(1-x)\}^l e^{(m-1)\log(1-x)}\}\, N(x)\, \mathrm{d}x.$$

Once again, since

$$\begin{aligned}&\int_0^1 (-x)^l \{e^{-mx} - e^{m\log(1-x)}\}\, N(x)\, \mathrm{d}x \\ &\quad=\; (-1)^l \int_0^1 x^l e^{-mx}(1 - \exp\{m[\log(1-x) + x]\})\, N(x)\, \mathrm{d}x \\ &\quad\sim\; (-1)^l \int_0^1 \tfrac{1}{2} m x^{l+2} e^{-mx} N(x)\, \mathrm{d}x \;\sim\; (-1)^l \frac{\Phi(m)}{m^{l+2}}\, \frac{(l+2)!}{2},\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 e^{m\log(1-x)}\{(-x)^l - \{\log(1-x)\}^l/(1-x)\}\, N(x)\, \mathrm{d}x \\ &\quad\sim\; -\int_0^1 (-x)^l \tfrac{1}{2}(l+2) x e^{-mx} N(x)\, \mathrm{d}x \\ &\quad\sim\; \tfrac{1}{2}(l+2)(-1)^{l+1}(l+1)!\, m^{-l-2}\Phi(m),\end{aligned}$$

it follows that $|D_l(m)| = o(m^{-l-2}\Phi(m))$ also. The lemma now follows by expressing the differences $\Phi_0^{(l)}(m) - \Phi^{(l)}(m)$ using the quantities $(D_j(m),\, 0 \leq j \leq l)$. □

In fact, the measure ν is an object in its own right: it is the Lévy measure of the *geometric* or *multiplicative* subordinator $\widehat{S}_t = 1 - \exp(-S_t)$. In terms of $\widehat{S}$ the composition $\mathcal{C}_n$ is defined as a record of occupancy counts for the gaps in the range of $\widehat{S}$ that are hit by at least one atom of a homogeneous Poisson process on $[0,1]$ with rate n.

2.3 Law of large numbers

For a law of large numbers, we begin by noting that

$$\mathbb{E}K_n \;=\; \mathbb{E}A_n(\infty) \;=\; \int_0^\infty \mathbb{E}\Phi(ne^{-S_t})\,\mathrm{d}t = \int_0^\infty \Phi(ne^{-s})U(\mathrm{d}s) \tag{2.10}$$

where U is the potential measure of S (*i.e.* $U[0,s]$ is the expected time S stays below s). Now, since Φ is slowly varying, it is plausible that

$$\mathbb{E}\Phi(ne^{-S_t}) \;\sim \Phi(ne^{-\mathbb{E}S_t}) \;=\; \Phi(ne^{-t}).$$

This motivates the introduction of

$$\Psi(n) \;:=\; \int_0^\infty \Phi(ne^{-t})\,\mathrm{d}t \;=\; \int_0^n \Phi(t)\,\frac{\mathrm{d}t}{t}\,, \tag{2.11}$$

as an approximation to K_n; under A1 it follows that $\Psi(n) \gg \log n$. Our aim in this section is to show that in fact $K_n \sim \Psi(n)$ for large n.

Lemma 2.4 *Assumption* A1 *implies* $\Psi(m) \gg \Phi(m)$.

Proof. This is a standard consequence of the uniform convergence theorem [4], which states that slow variation implies $\Phi(mu)/\Phi(m) \to 1$, uniformly in u bounded away from 0 and ∞. $\Box$

It hence follows that, for any fixed T, and as $n \to \infty$,

$$\frac{A_n(T)}{\Psi(n)} \;<\; \frac{T\Phi(n)}{\Psi(n)} \;\to\; 0 \quad \text{a.s.} \tag{2.12}$$

and also that

$$\frac{\int_0^T \Phi(ne^{-t})\,\mathrm{d}t}{\Psi(n)} \;<\; \frac{T\Phi(n)}{\Psi(n)} \;\to\; 0. \tag{2.13}$$

The convergence in (2.12) and (2.13) also holds if the fixed time T is replaced by an a.s. finite random time τ which is measurable with respect to $\sigma\{S_t,\, t \ge 0\}$.

The next lemma explores the error caused by replacing U with the Lebesgue measure in (2.10).

Lemma 2.5 *For an arbitrary subordinator S with $\mathbb{E}S_1^2 < \infty$,*

$$\Psi(n) \le \mathbb{E}\,K_n \le \Psi(n) + \lambda\Phi(n) \tag{2.14}$$

for some constant $\lambda > 0$. It follows that, under A1, *$\mathbb{E}K_n/\Psi(n) \to 1$ as $n \to \infty$.*

Proof. Let X be a random variable with density $N_0(x)$, independent of S and Π_n. The closed range $X+\mathcal{R}$ of the delayed subordinator $(X+S_t, t \ge 0)$ is a stationary counterpart of $\mathcal{R}$. Ignoring the interval $]0,X[$, the expected number of occupied gaps produced by $X+S_t$ is precisely

$$\mathbb{E}K_{ne^{-X}} = \Psi(n)$$

because, by stationarity, the potential measure of $S_t + X$ is Lebesgue measure. Since K_n is nondecreasing in n, we have

$$K_n \geq_d K_{ne^{-X}},$$

and thence $\mathbb{E}K_n \geq \Psi(n)$. This inequality can also be argued analytically, by using $U[0,s] \geq s$ and the monotonicity of Φ.

Let τ be the passage time for S through X. Because S has a nontrivial overshoot over X (while $X+S$ has none, since $X+S_0 = X$), the number of occupied gaps within $[S_\tau, \infty]$ produced by S is stochastically smaller than $K_{ne^{-X}}$ (produced by $X+S$). It follows that

$$K_n - \mathcal{K}_n(\tau) <_d K_{ne^{-X}}\,;$$

passing to expectations, we obtain

$$\mathbb{E}K_n \leq \mathbb{E}K_{ne^{-X}} + \mathbb{E}\mathcal{K}_n(\tau) = \Psi(n) + \mathbb{E}\int_0^\tau \Phi(ne^{-S_t})\mathrm{d}t \leq \Psi(n) + \Phi(n)\mathbb{E}\tau\,.$$

Now, $\mathbb{E}\tau < \infty$ follows from $\mathbb{E}X < \infty$, which is implied by the assumption $\mathbb{E}S_1^2 < \infty$. This completes the proof of (2.14), with $\lambda = \mathbb{E}\tau$. The convergence of $\mathbb{E}K_n/\Psi(n)$ is now a consequence of Lemma 2.4. □

We now show that K_n is close to $A_n(\infty)$.

Lemma 2.6 *Under Assumption* A1

$$\mathbb{E}(K_n - A_n(\infty))^2 \ \sim\ \Psi(n).$$

Furthermore, considering the whole path of $\mathcal{K}_n - A_n$, we have

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{0\leq T\leq\infty} |\mathcal{K}_n(T) - A_n(T)| > b_n\Big] \ =\ 0,$$

for all sequences b_n such that $b_n^{-2}\,\Psi(n) \to 0$.

Proof. The difference $\mathcal{K}_n(T) - A_n(T)$ is a square-integrable martingale of locally bounded variation with respect to the filtration $\mathcal{F}_{T,n}$, and has all jumps of size 1; this yields the formula [12, Section 15.2]

$$\mathbb{E}(K_n - A_n(\infty))^2 \ =\ \mathbb{E}A_n(\infty) \ \sim\ \Psi(n), \tag{2.15}$$

proving the first part. The second follows from Kolmogorov's inequality, which gives

$$\mathbb{P}\Big[\sup_{0\leq T\leq\infty} |\mathcal{K}_n(T) - A_n(T)| > b_n\Big] \ \leq\ b_n^{-2}\,\mathbb{E}A_n(\infty). \quad \square \tag{2.16}$$

Next is the law of large numbers for the compensator.

Proposition 2.7 *Under Assumption* A1, *as $n \to \infty$, $A_n(\infty)/\Psi(n) \to 1$ almost surely and in the mean.*

Proof. Fix ε, and define

$$\tau \ := \ \sup\{t \geq 0 : \ t^{-1}|S_t - t| > \varepsilon\},$$

finite almost surely. By the monotonicity of Φ,

$$(1+\varepsilon)^{-1} \int_{(1+\varepsilon)\tau}^{\infty} \Phi(ne^{-s}) \, \mathrm{d}s \ = \ \int_{\tau}^{\infty} \Phi(ne^{-(1+\varepsilon)t}) \, \mathrm{d}t \ < \ \int_{\tau}^{\infty} \Phi(ne^{-S_t}) \, \mathrm{d}t$$
$$< \ \int_{\tau}^{\infty} \Phi(ne^{-(1-\varepsilon)t}) \, \mathrm{d}t \ = \ (1-\varepsilon)^{-1} \int_{(1-\varepsilon)\tau}^{\infty} \Phi(ne^{-s}) \, \mathrm{d}s.$$

Dividing by $\Psi(n)$ and using (2.12) and (2.13), we make the sandwich

$$\frac{1}{1+\varepsilon} \ < \ \liminf_{n\to\infty} \frac{A_n(\infty)}{\Psi(n)} \ \leq \ \limsup_{n\to\infty} \frac{A_n(\infty)}{\Psi(n)} \ < \ \frac{1}{1-\varepsilon} \quad \text{a.s.},$$

and now let $\varepsilon \to 0$ to obtain almost sure convergence. Convergence in the mean then follows from $\mathbb{E}K_n = \mathbb{E}A_n(\infty)$ and Lemma 2.5, together with the fact that $A_n(\infty)/\Psi(n) \geq 0$ a.s. $\square$

Finally, we have all ingredients to establish the law of large numbers for K_n.

Theorem 2.8 *As $n \to \infty$, $K_n/\Psi(n) \to 1$ almost surely and in the mean.*

Proof. It follows from Lemma 2.6 that

$$\mathbb{E}\left(\frac{K_n}{\Psi(n)} - \frac{A_n(\infty)}{\Psi(n)}\right)^2 \ = \ \frac{1}{\Psi(n)} \, \mathbb{E}\left(\frac{A_n(\infty)}{\Psi(n)}\right) \ \sim \ \frac{1}{\Psi(n)}\,,$$

and convergence in the mean follows from Proposition 2.7. Then, because $\Psi(n)$ is increasing, continuous and unbounded, we can select n_j to satisfy $\Psi(n_j) = j^2$. Then from

$$\sum_{j=1}^{\infty} \left(\frac{K_{n_j}}{\Psi(n_j)} - \frac{A_{n_j}(\infty)}{\Psi(n_j)}\right)^2 < \infty$$

we conclude, in a standard way, that $K_{n_j}/\Psi(n_j) \to 1$ a.s. along the subsequence. Now, because both K_n and $\Psi(n)$ are increasing in n, the inequalities

$$\frac{K_{n_j}}{\Psi(n_{j+1})} \leq \frac{K_n}{\Psi(n)} \leq \frac{K_{n_{j+1}}}{\Psi(n_j)}$$

hold for $n_j \leq n \leq n_{j+1}$. The convergence almost surely follows from these relations and the trivial fact that $\Psi(n_j)/\Psi(n_{j+1}) \to 1$. $\square$

Along the same lines, $K_n \gg \mathcal{K}_n(T)$ for each fixed T. This property can be shown to be characteristic of the slow variation case.

2.4 The variance

The aim of this section is to derive asymptotics of the variance of K_n and $A_n(\infty)$. We start with the explicit formula

$$\operatorname{Var} A_n(\infty) = 2 \int_0^\infty \Phi(ne^{-s})U(\mathrm{d}s) \int_0^\infty \Phi(ne^{-s-v})\mathrm{d}_v\{U[0,v] - U[s,s+v]\}, \tag{2.17}$$

where d_v indicates the active variable of integration. The formula is derived by using the representation of the path past t as $(S_u, u \geq t) =_d (S_t + S'_u, u \geq 0)$ with S' an independent copy of S, and using a familiar symmetrisation trick for squared integrals:

$$\begin{aligned}
\operatorname{Var} A_n(\infty) &= \mathbb{E}\left(\int_0^\infty \Phi(ne^{-S_t})\mathrm{d}t\right)^2 - \left(\int_0^\infty \Phi(ne^{-s})\, U(\mathrm{d}s)\right)^2 \\
&= 2\,\mathbb{E}\left(\int_0^\infty \Phi(ne^{-S_t})\,\mathrm{d}t \int_t^\infty \Phi(ne^{-S_u})\,\mathrm{d}u\right) \\
&\qquad -2\int_0^\infty \Phi(ne^{-s})\,U(\mathrm{d}s)\int_s^\infty \Phi(ne^{-v})\,U(\mathrm{d}v) \\
&= 2\,\mathbb{E}\left(\int_0^\infty \Phi(ne^{-S_t})\,\mathrm{d}t \int_0^\infty \Phi(ne^{-S_t-S'_u})\,\mathrm{d}u\right) \\
&\qquad -2\int_0^\infty \Phi(ne^{-s})\,U(\mathrm{d}s)\int_0^\infty \Phi(ne^{-s-v})\,U(s+\mathrm{d}v) \\
&= 2\int_0^\infty \Phi(ne^{-s})\,U(\mathrm{d}s)\int_0^\infty \Phi(ne^{-s-v})\,U(\mathrm{d}v) \\
&\qquad -2\int_0^\infty \Phi(ne^{-s})\,U(\mathrm{d}s)\int_0^\infty \Phi(ne^{-s-v})\,U(s+\mathrm{d}v) \\
&= 2\int_0^\infty \Phi(ne^{-s})U(\mathrm{d}s)\int_0^\infty \Phi(ne^{-s-v})\mathrm{d}_v\{U[0,v]-U[s,s+v]\}.
\end{aligned}$$

Next is a more informative asymptotic formula, very much in the spirit of the asymptotics for the expectation $\mathbb{E}K_n \sim \Psi(n)$ derived before. To state it, we first define

$$\Psi_2(n) \;:=\; \int_0^\infty \Phi^2(ne^{-s})\,\mathrm{d}s \;=\; \int_0^n \Phi^2(s)\,\frac{\mathrm{d}s}{s}\,. \tag{2.18}$$

Lemma 2.9 *For a subordinator such that Φ is slowly varying at infinity,*

$$\operatorname{Var} A_n(\infty) \;\sim\; \sigma^2\Psi_2(n)\,. \tag{2.19}$$

If also A1 *holds, then the same asymptotics hold for K_n;*

$$\operatorname{Var} K_n \;\sim\; \operatorname{Var} A_n(\infty) \;\sim\; \sigma^2\Psi_2(n)\,.$$

Proof. Renewal theory tells us that

$$0 \;\leq\; U[0,s] - s \;\to\; \sigma^2/2\,,$$

where the constant appears as the mean value of the delay variable X:

$$\mathbb{E}X = \int_0^\infty x N_0(x)\,\mathrm{d}x = \frac{1}{2}\int_0^\infty x^2\,\nu_0(\mathrm{d}x) = \frac{1}{2}\mathrm{Var}\,S_1 = \frac{\sigma^2}{2}$$

(the third equality follows from (2.1)). This general fact holds for arbitrary square-integrable subordinators, and it follows easily from the compound Poisson case treated in [6].

We can therefore, for ε given, select s_0 and v_0 so large that, for $s > s_0$ and $v > v_0$,

$$|U[0,v_0] - U[s,s+v_0] - \sigma^2/2| < \varepsilon\,; \qquad |U[v_0,v] - U[s+v_0,s+v]| < \varepsilon\,. \tag{2.20}$$

Now, writing $G_s(v) = U[0,v] - U[s,s+v]$, we have

$$G_s(0) \;=\; 0, \quad \|G_s\| \;:=\; \sup_{v>0}|G_s(v)| \;\le\; \sigma^2 \quad \text{and} \quad |G_s(v) - \sigma^2/2| \;<\; \varepsilon$$

for $s > s_0$ and $v > v_0$. Hence, by partial integration, the inner integral is at most $\|G_s\|\Phi(ne^{-s})$, so that truncating the external integral in (2.17) at the lower bound s_0 yields an error of at most $\lambda s_0 \Phi^2(n)$, which is negligible when compared with the claimed asymptotics. Similarly, truncating the external integral at an upper bound $\log n - \psi$ yields an error of at most

$$2\sigma^2\int_{\log n-\psi}^\infty \Phi^2(ne^{-s})\,U(\mathrm{d}s) \;\le\; 2e^{2\psi}\sigma^2\left\{\int_0^\infty e^{-2v}\,\mathrm{d}v + U^*\right\} \;=\; 2e^{2\psi}\sigma^2(1+U^*),$$

using (2.6), where $U^* := \sup_{s>0}|U(s) - s|$. Then truncating the internal integral at the upper bound v_0 results in an error estimated as

$$2\int_{s_0}^\infty \Phi(ne^{-s})\,U(\mathrm{d}s)\int_{v_0}^\infty \Phi(ne^{-s-v})\,\mathrm{d}\{U[v_0,v] - U[s+v_0,s+v]\} < 2\varepsilon\int_{s_0}^\infty \Phi^2(ne^{-s})\,U(\mathrm{d}s), \tag{2.21}$$

while, as above,

$$\left|\int_0^\infty \Phi^2(ne^{-s})(U(\mathrm{d}s) - \mathrm{d}s)\right| \;\le\; 2U^*\Phi^2(n),$$

so that

$$\int_0^\infty \Phi^2(ne^{-s})\,U(\mathrm{d}s) \sim \int_0^n \Phi^2(s)\,\frac{\mathrm{d}s}{s}. \tag{2.22}$$

Thus we are reduced to evaluating

$$J_{\varepsilon,n} := 2\int_{s_0}^{\log n-\psi_n} \Phi(ne^{-s})\,U(\mathrm{d}s)\int_0^{v_0} \Phi(ne^{-s-v})\,\mathrm{d}\{U[0,v] - U[s,s+v]\},$$

where we let $\psi = \psi_n \to \infty$ slowly enough that

$$e^{2\psi_n} \;\ll\; \int_0^n \Phi^2(ne^{-s})\,U(\mathrm{d}s).$$

But in the range $v < v_0$, $s < \log n - \psi_n$ we have e^{-v} bounded from 0 and ∞, and $ne^{-s} > e^{\psi_n} \to \infty$; hence, by the uniform convergence theorem for slowly varying functions [4],

$$\frac{\Phi(ne^{-s-v})}{\Phi(ne^{-s})} \to 1\,, \qquad \text{as } \ n \to \infty,$$

uniformly in such s and v. With this substitution and using (2.20) and (2.22) we obtain

$$J_{\varepsilon,n} \sim (1+O(\varepsilon))\sigma^2 \int_{s_0}^{\infty} \Phi^2(ne^{-s})\, U(\mathrm{d}s) \ \sim (1+O(\varepsilon))\sigma^2 \int_0^n \Phi^2(s)\, \frac{\mathrm{d}s}{s}\,.$$

Hence, recalling (2.21) and sending $\varepsilon \to 0$, the desired asymptotics follow.

Noting that, under A1,

$$\Psi_2(n) \ = \ \int_0^n \Phi^2(s)\, \frac{\mathrm{d}s}{s} \ \gg \ \int_0^n \Phi(s)\, \frac{\mathrm{d}s}{s} \ = \ \Psi(n), \tag{2.23}$$

it follows from Lemma 2.6 that the asymptotics of $\operatorname{Var} K_n$ are implied by those of $\operatorname{Var} A_n(\infty)$. □

Remarks. A general subordinator $\widetilde{S}$ with

$$\mathtt{m} \ := \ \mathbb{E}\widetilde{S}_1 \ = \ \int_0^\infty x\, \tilde{\nu}_0(\mathrm{d}x) \quad \text{and} \quad \tau^2 \ := \ \operatorname{Var} \widetilde{S}_1 \ = \ \int_0^\infty x^2\, \tilde{\nu}_0(\mathrm{d}x)$$

yields a subordinator S with $\mathbb{E}S_1 = 1$, by scaling time so that $S_t = \widetilde{S}_{t/\mathtt{m}}$; S has $\Phi = \mathtt{m}^{-1}\widetilde{\Phi}$ and $\sigma^2 = \mathtt{m}^{-1}\tau^2$, and has the same quantities K_n and $A_n(\infty)$ as $\widetilde{S}$. Hence, for $\widetilde{S}$, we have

$$\mathbb{E}A_n(\infty) \ \sim \ \int_0^n s^{-1}\Phi(s)\, \mathrm{d}s \ = \ \mathtt{m}^{-1} \int_0^n s^{-1}\widetilde{\Phi}(s)\, \mathrm{d}s$$

and

$$\operatorname{Var} A_n(\infty) \ \sim \ \sigma^2 \int_0^n s^{-1}\Phi^2(s)\, \mathrm{d}s \ = \ \tau^2\mathtt{m}^{-3} \int_0^n s^{-1}\widetilde{\Phi}^2(s)\, \mathrm{d}s\,.$$

For the gamma subordinator with $\tilde{\nu}_0(\mathrm{d}x) = ax^{-1}e^{-\theta x}\,\mathrm{d}x$, we have $\widetilde{\Phi}_0(s) = a\log(1+s/\theta)$, $\mathtt{m} = a/\theta$ and $\tau^2 = a/\theta^2$. Hence

$$\operatorname{Var} K_n \ \sim \ \operatorname{Var} A_n(\infty) \ \sim \ \theta \int_0^n s^{-1}\log^2(1+s/\theta)\, \mathrm{d}s \ \sim \ \frac{\theta}{3}\log^3 n,$$

agreeing with the asymptotics for gamma-like subordinators obtained in [11] by a method based on the Mellin transform.

In the compound Poisson case, the asymptotics of $\operatorname{Var} A_n(\infty)$ and $\operatorname{Var} K_n$ are different, because $\Psi(n)$ is no longer of smaller order than $\Psi_2(n)$ as in (2.23). Instead, with the normalisation $\nu_0[0,\infty] = 1$, so that $\lim_{n\to\infty} \widetilde{\Phi}(n) = 1$, we have

$$\operatorname{Var} A_n(\infty) \ \sim \ \frac{\tau^2}{\mathtt{m}^3}\log n\,; \qquad \operatorname{Var} K_n \ \sim \ \frac{\tau^2 - \mathtt{m}^2}{\mathtt{m}^3}\log n,$$

(see [7]), so that (2.19) is valid only for the compensator.

In the case of regular variation, $\Psi_2(n)$ still gives the correct order of growth for the variances of both quantities, but the coefficients are not as in (2.19); see [10] for details.

It is immediate from the definitions (2.11) and (2.18) that $\liminf_{n\to\infty} \Psi_2(n)/\Psi(n) \geq \Phi(M)$ for all $M > 0$, and hence that $\Psi_2(n) \gg \Psi(n)$ as $n \to \infty$. In consequence, from (2.19) and Lemma 2.6, the fluctuations of the process $\mathcal{K}_n - A_n$ are of smaller order than those of A_n, so that, when studying limit theorems for $\mathcal{K}_n(t)$, it is enough to consider $A_n(t)$.

3 The key assumption

The asymptotics of moments only required the monotonicity of Φ and the property of slow variation. In order to progress to a finer description of the asymptotics of K_n, we need a further assumption in addition to A1. To express it, we begin by associating with Φ the function

$$L(s) \;:=\; \frac{\Phi(s)}{s\Phi'(s)}\,,$$

so that $(sL(s))^{-1} = (\log \Phi(s))'$ and

$$\Phi(s) \;=\; \Phi(1)\exp\left\{\int_1^s \frac{\mathrm{d}z}{zL(z)}\right\}. \tag{3.24}$$

Assumption A1 forces $\lim_{s\to\infty} L(s) = \infty$, because the last formula is just an instance of the Karamata representation for slowly varying functions [4]. Constantly keep in mind that the faster L, the slower Φ.

Our extra assumption on Φ is expressed via L, and puts a limit on the way in which it can vary locally: we assume that there exist $s_0 \geq 1$ and $k > 0$ such that

$$\textbf{Assumption A2}: \qquad \left|\frac{sL'(s)}{L(s)}\right| \;<\; \frac{k}{\log s} \quad \text{for all} \quad s \geq s_0. \tag{3.25}$$

Because the right side in (3.25) goes to zero with s, the function L is itself slowly varying; under A1 the latter property is equivalent to the slow variation of $s\Phi'(s)$.

Lemma 3.1 *If* A1 *holds and* L *is slowly varying, then for* $L_0(s) := \Phi_0(s)/(s\Phi_0'(s))$

$$\left|\frac{sL'(s)}{L(s)} - \frac{sL_0'(s)}{L_0(s)}\right| = o(s^{-1}L^2(s));$$

thus Assumption A2 *can equivalently be stated using* L_0 *in place of* L.

Proof. Direct calculation shows that

$$\frac{sL'(s)}{L(s)} - \frac{sL_0'(s)}{L_0(s)} = s\left\{\left(\frac{\Phi'(s)}{\Phi(s)} - \frac{\Phi_0'(s)}{\Phi_0(s)}\right) - \left(\frac{\Phi''(s)}{\Phi'(s)} - \frac{\Phi_0''(s)}{\Phi_0'(s)}\right)\right\}.$$

Now we apply Lemma 2.3 to bound differences between the derivatives of Φ_0 and those of Φ, and (2.9) to bound the derivatives themselves, and we also note that $\Phi'(s) = \Phi(s)/sL(s)$. The lemma follows. $\square$

We note in passing that both functions $1/L_0$ and $1/L$ can be given various probabilistic interpretations. For instance, in the spirit of (2.1),

$$1/L_0(n) = \mathbb{E}\exp\{-n(S_\tau - \xi)\}$$

where ξ is an independent exponential level with rate n and τ is the passage time across ξ, so that $S_\tau - \xi$ is the overshoot at ξ, see [15, Corollary 1 (ii)]. The function $1/L$ determines a conditional rate for creating singleton blocks of $\mathcal{C}_n$, meaning that $1/L(s)$, with $s = ne^{-S_t}$, is the conditional probability that a jump of S at time t covers exactly one Poisson point given at least one point is covered.

Although we regard A2 as a *local* condition, under circumstances it can restrict the *global* growth of L. For suppose that L is eventually increasing. Then, introducing $h(m) := L(m)/(mL'(m))$, we have in the usual way

$$L(m) = L(1)\,\exp\left(\int_1^m \frac{\mathrm{d}z}{zh(z)}\right).$$

Now, because eventually $L' \geq 0$, Assumption A2 reads as $h(m) > k^{-1}\log m$, hence yielding a global bound $L(m) < \lambda \log^k m$. In the other direction, observe that, even if L is not monotone, the inequality inverse to (3.25), $L(m) > \lambda \log^k m$ with some $k > 1$, would disagree with A1, because in this case Φ would be bounded.

Remark. Of course, when (3.25) holds for some s_0 and k, we can set $s_0 = 1$ by taking k sufficiently large. This will suffice for our purposes, but gives a poor idea of the growth of L.

Assumption A2 implies that

$$\left(\frac{\log y}{\log x}\right)^{-k} < \frac{L(y)}{L(x)} < \left(\frac{\log y}{\log x}\right)^{k}, \qquad s_0 < x < y\,. \tag{3.26}$$

To see this, for $s_0 < x < y$, observe that

$$\frac{L(y)}{L(x)} = \exp\left\{\int_x^y \frac{\mathrm{d}z}{h(z)\,z}\right\} < \exp\left\{\int_x^y \frac{\mathrm{d}z}{z}\left(\frac{k}{\log z}\right)\right\} = \left(\frac{\log y}{\log x}\right)^{k},$$

and similarly that

$$\frac{L(y)}{L(x)} > \exp\left\{-\int_x^y \frac{\mathrm{d}z}{z}\left(\frac{k}{\log z}\right)\right\} = \left(\frac{\log y}{\log x}\right)^{-k}.$$

Assumption A2 is a kind of 'second order' slow variation, in the sense that the function $\log\Phi$ has a form of de Haan's property (see [4, Section 3.0]):

$$\frac{\log\Phi(ne^{-s}) - \log\Phi(n)}{1/L(n)} \to -s\,, \qquad (n \to \infty).$$

However, A2 is stronger than just this, and offers a better control on the variability of Φ; in particular we have the following estimate for the remainder.

Lemma 3.2 *Under assumptions* A1–A2*, we have*

$$\left|\log\left\{\frac{\Phi(me^{-s})}{\Phi(m)}\right\} + \frac{s}{L(m)}\right| \;<\; \frac{\varkappa s^2}{L(m)\log m}\,, \tag{3.27}$$

for all $m \geq 1$ *and* $|s| < \frac{1}{2}\log m$*, where* $\varkappa = k2^k$.

Proof. By Taylor's formula with the remainder in Lagrange's form,

$$\log\Phi(me^{-s}) \;=\; \log\Phi(m) - \frac{s}{L(m)} - \frac{s^2}{2}\frac{m^*L'(m^*)}{L(m^*)^2},$$

for some m^* such that

$$m \wedge (me^{-s}) \leq m^* \leq m \vee (me^{-s})\,.$$

Since $|s| < (\log m)/2$, we have $m^{1/2} \leq m^* \leq m^{3/2}$, and hence $\frac{1}{2}\log m \leq \log m^* \leq \frac{3}{2}\log m$; and also, from A2,

$$\left|\frac{m^*L'(m^*)}{L(m^*)}\right| \;<\; \frac{k}{\log m^*} \;\leq\; \frac{2k}{\log m}\,.$$

On the other hand, for m^* in this range, (3.26) implies that $L(m^*) > 2^{-k}L(m)$. Hence

$$\frac{m^*L'(m^*)}{2L(m^*)^2} \;<\; \frac{k2^k}{L(m)\log m}\,,$$

as required. □

Remark. So, loosely speaking, we are dealing with functions L that grow slowly enough, satisfying $L(n^{1/2}) \asymp L(n)$. Indeed, it can be shown that $L(n) \gg L(n^{1/2})$ implies the convergence

$$\int_1^\infty \frac{\mathrm{d}s}{sL(s)} \;<\; \infty\,,$$

in which case the Lévy measure is finite.

Corollary 3.3 *Under Assumptions* A1–A2*, we have*

$$\Phi(n)\,e^{-3s/2L(n)} \;\leq\; \Phi(ne^{-s}) \;\leq\; \Phi(n)\,e^{-s/2L(n)}$$

for $0 \leq s \leq \frac{1}{2(\varkappa\vee 1)}\log n$.

Proof. Immediate from the above. □

Corollary 3.4 *Under Assumptions* A1–A2*, if* $L(n) < \frac{1}{4(\varkappa\vee 1)}\log n$*, we have*

$$\Psi(n) \;\leq\; 1 + \Phi(n)L(n)\left(2 + \tfrac{4(\varkappa\vee 1)}{e}\right);$$
$$\tfrac{1}{3}(1-e^{-6})\Phi^2(n)L(n) \;\leq\; \Psi_2(n) \;\leq\; \tfrac{1}{2} + \Phi^2(n)L(n)\{1 + \tfrac{2(\varkappa\vee 1)}{e}\}.$$

If $L(n) \geq \frac{1}{4(\varkappa\vee 1)}\log n$*, we have*

$$\Psi(n) \;\leq\; 1 + \Phi(n)\log n\left(\tfrac{1}{2(\varkappa\vee 1)} + \tfrac{1}{e}\right);$$
$$\tfrac{1}{3}(1-e^{-6})\Phi^2(n)\log n \;\leq\; \Psi_2(n) \;\leq\; \tfrac{1}{2} + \Phi^2(n)\log n\{\tfrac{1}{2(\varkappa\vee 1)} + \tfrac{1}{e^2}\}.$$

Proof. Write $k_n := \frac{1}{2(\varkappa \vee 1)} \log n$. From the upper bound in Corollary 3.3, we have

$$\int_0^{k_n} \Phi(ne^{-t})\,\mathrm{d}t \;\le\; \Phi(n)\{2L(n) \wedge k_n\},$$

and

$$\begin{aligned}\int_{k_n}^{\log n} \Phi(ne^{-t})\,\mathrm{d}t &\le \Phi(n)e^{-k_n/2L(n)}\log n \\ &= 4(\varkappa \vee 1)\Phi(n)L(n)\left\{\frac{k_n}{2L(n)}\exp\left(-\frac{k_n}{2L(n)}\right)\right\};\end{aligned}$$

furthermore, from (2.6),

$$\int_{\log n}^{\infty} \Phi(ne^{-t})\,\mathrm{d}t \;\le\; \int_{\log n}^{\infty} ne^{-t}\,\mathrm{d}t \;=\; 1.$$

The bounds for $\Psi(n)$ now follow from its definition, and because $xe^{-x} \le e^{-1}$ for $x \ge 0$. The proof of the upper bounds for $\Psi_2(n)$ is analogous.

For the lower bound on $\Psi_2(n)$, integrate $\Phi^2(ne^{-t})$ from 0 to $\min\{k_n, 2L(n)\}$, and then use the lower bound in Corollary 3.3. $\square$

For the rest of this paper both assumptions A1 and A2 will be taken for granted, even if not explicitly mentioned.

4 The forward argument

In this section, under a wide range of circumstances in which $L(n) = O(\log n)$, we show that the quantity

$$A_n^*(T) := \int_0^{T\wedge \log n} \Phi(ne^{-t})\left(1 - \frac{S_t - t}{L(ne^{-t})}\right)\mathrm{d}t \tag{4.28}$$

is an adequate approximation to $A_n(T \wedge \tau_n)$, and hence, in view of Lemma 2.2, to $A_n(T)$. This is a very attractive result, because the random process S appears only linearly in $A_n^*(T)$, making it easier to determine the approximate behaviour of $A_n(T)$ from knowledge of that of S. The way that the approximation is proved is to show that the quantity $\sup_{T\ge 0}|A_n(T\wedge\tau_n) - A_n^*(T)|$ is asymptotically smaller than the scale of fluctuations of A_n. In view of Lemma 2.2 and Corollary 3.4, in order to achieve this when $L(n) = O(\log n)$, we need to prove that, with probability tending to 1,

$$\sup_{T\ge 0}|A_n(T\wedge\tau_n) - A_n^*(T)| = o(\Phi(n)\sqrt{L(n)}).$$

To this end, we define the centred process

$$Z_t := S_t - t$$

and restrict attention as far as possible to realisations of S for which the paths of Z are reasonably nice. This we make precise as follows. First, for any $T, \varphi, \psi > 0$, we define the events

$$\begin{aligned} B_0(n) &:= \{\tfrac{1}{2}\log n \le \tau_n \le 2\log n\}; && (4.29)\\ B_1(T,\varphi) &:= \Bigl\{\sup_{0\le t\le T}(t\vee 1)^{-1/2}|Z_t| \le \varphi\Bigr\}; && (4.30)\\ B_2(\psi) &:= \Bigl\{\sup_{t\ge\psi} 2t^{-1}|Z_t| \le 1\Bigr\}. && (4.31)\end{aligned}$$

The paths of Z are well behaved if $B_1(T,\varphi)$ holds for φ not too large and for large enough T, and if $B_2(\psi)$ holds for ψ not too large. With reference to these desiderata, we have the following lemma.

Lemma 4.1 *For $T, \varphi, \psi > 0$, we have*

$$\begin{aligned} \mathbb{P}[B_0^c(n)] &\le 8\sigma^2/\log n; && (4.32)\\ \mathbb{P}[B_1^c(T,\varphi)] &\le 2\sigma^2\varphi^{-2}\lceil\log_2 T\rceil \qquad (T>2); && (4.33)\\ \mathbb{P}[B_2^c(\psi)] &\le 32\sigma^2\psi^{-1}, && (4.34)\end{aligned}$$

and also $B_0(n) \supset B_1(2\log n, \frac{1}{2}\log^\alpha n)$ for any $0 \le \alpha \le \frac{1}{2}$ and $n \ge 3$.

Proof. First, by Kolmogorov's inequality for the centred, independent increments process Z, we have

$$\mathbb{P}[B_0^c(n)] \le \mathbb{P}\Bigl[\sup_{0\le u\le 2\log n}|Z_u| > \tfrac{1}{2}\log n\Bigr] \le 8\sigma^2/\log n.$$

The remaining statements are proved by combining Kolmogorov's inequality with geometric dissection, in a rather standard fashion. For the second inequality, we have

$$\begin{aligned} \mathbb{P}[B_1^c(T,\varphi)] &\le \sum_{r=1}^{\lceil\log_2 T\rceil}\mathbb{P}\left[\max_{0\le t\le 2^r}(2^{r-1})^{-1/2}|Z_t| > \varphi\right]\\ &\le \sum_{r=1}^{\lceil\log_2 T\rceil}\frac{2^r\sigma^2}{\varphi^2 2^{r-1}} = 2\sigma^2\varphi^{-2}\lceil\log_2 T\rceil.\end{aligned}$$

For the third, we have

$$\begin{aligned} \mathbb{P}[B_2^c(\psi)] &\le \sum_{r\ge\lceil\log_2\psi\rceil}\mathbb{P}\left[\max_{0\le t\le 2^r}2|Z_t| > 2^{r-1}\right]\\ &\le \sum_{r\ge\lceil\log_2\psi\rceil}16\sigma^2 2^{-r} \le 32\sigma^2\psi^{-1}.\end{aligned}$$

□

The following corollary needs no proof.

Corollary 4.2 *For any positive sequences T_n, φ_n, ψ_n we have*

$$\begin{aligned} &\text{(i)} \qquad \lim_{n\to\infty} \mathbb{P}[B_1(T_n,\varphi_n)] = 1 \quad \textit{if} \quad \varphi_n^{-2}\log T_n \to 0; \\ &\text{(ii)} \qquad \lim_{n\to\infty} \mathbb{P}[B_2(\psi_n)] = 1 \quad \textit{if} \quad \psi_n \to \infty. \end{aligned}$$

For the further argument, we distinguish two cases, relating to the global pattern of growth of $\Phi(n)$, each of which needs separate treatment. The idea of the distinction can be seen from the following formula for the variance of the linearised compensator when $T < \log n$:

$$\operatorname{Var} A_n^*(T) = \sigma^2 \int_0^T \{\Phi(ne^{-t}) - \Phi(ne^{-T})\}^2 \,\mathrm{d}t\,, \tag{4.35}$$

which is derived by writing (4.28) for the centred A_n^* as a stochastic integral:

$$A_n^*(T) - \mathbb{E}A_n^*(T) = -\int_0^T \{\Phi(ne^{-t}) - \Phi(ne^{-T})\}\,\mathrm{d}(S_t - t)\,,$$

and using the independence of increments. So, when Φ is a function like a power of logarithm, the difference $\Phi(ne^{-t}) - \Phi(ne^{-T})$ is of constant order over the whole time-range from 0 to $\log n$. On the other hand, if Φ grows fast enough, the first term will dominate, and the principal contribution to the integral will come from times $t = o(\log n)$, as is also the case if Φ is regularly varying.

4.1 Moderately growing Φ

We begin with the boundary case, which includes the gamma-like subordinators [11], when $\Phi(n)$ grows more or less like a power of $\log n$. Here, all times t between 0 and $\log n$ contribute more or less evenly to the fluctuations of A_n. This case is defined by a global condition on the function L; that, for some $1 \le c_2 < \infty$ and for some m_0, and with $c_1 := \{3(\varkappa \vee 1)\}^{-1}$,

$$\frac{c_1 \log m}{6 \log\log m} \;\le\; L(m) \;\le\; c_2 \log m, \quad m \ge m_0. \tag{4.36}$$

The next lemma is a preliminary to proving that, under these circumstances, A_n^* is a good approximation to A_n. It enables us to truncate the integrals defining $A_n(T)$ and $A_n^*(T \wedge \tau_n)$ close to $\log n$, when the paths of Z are nice enough.

Lemma 4.3 *On the event $B_1(2\log n, \varphi_n)$, and for $0 < \psi_n \le \log n - \varphi_n\sqrt{\log n}$, we have, for all $T > 0$,*

$$0 \;\le\; \int_0^{\tau_n \wedge T} \Phi(ne^{-S_t})\,\mathrm{d}t - \int_0^{\psi_n \wedge T} \Phi(ne^{-S_t})\,\mathrm{d}t \;\le\; \eta_n; \tag{4.37}$$

$$\int_{\psi_n \wedge T}^{T \wedge \log n} \Phi(ne^{-t}) \left|1 - \frac{Z_t}{L(ne^{-t})}\right| \,\mathrm{d}t \;\le\; \eta_n, \tag{4.38}$$

where

$$\eta_n = (\log n - \psi_n + \varphi_n\sqrt{2\log n})\Phi(n\exp\{-\psi_n + \varphi_n\sqrt{\psi_n}\}).$$

Proof. On $B_1(2\log n, \varphi_n)$, we have

$$\log n - \varphi_n\sqrt{\log n} \;\le\; \tau_n \;\le\; \log n + \varphi_n\sqrt{2\log n},$$

and so $\psi_n \le \min\{\tau_n, \log n\}$. Hence (4.37) and (4.38) are both zero if $T \le \psi_n$. The first part of the lemma then merely uses the fact that

$$\psi_n \le \tau_n \le \log n + \varphi_n\sqrt{2\log n},$$

combined with the largest possible value of the integrand in this range. For the second part, recall (5.54) so that

$$\int_{\psi_n}^{\log n} \Phi(ne^{-t})\,\mathrm{d}t \;\le\; \Phi(ne^{-\psi_n})(\log n - \psi_n)$$

and from (4.30)

$$\int_{\psi_n}^{\log n} \Phi(ne^{-t})\frac{|Z_t|}{L(ne^{-t})}\,\mathrm{d}t \;\le\; \Phi(ne^{-\psi_n})\varphi_n\sqrt{\log n}. \qquad \square$$

It follows from A2, (4.36) and the definition of L that

$$\Phi(ne^{-t}) \;=\; \Phi(n)\exp\left\{-\int_{ne^{-t}}^{n}\frac{dy}{yL(y)}\right\} \;\le\; \Phi(n)\left\{1-\frac{t}{\log n}\right\}^{1/c_2} \tag{4.39}$$

for all n and t such that $ne^{-t} \ge m_0$. Thus, taking

$$\psi_n = \log n - 2u_n\sqrt{\log n}, \tag{4.40}$$

for $u_n \ge \varphi_n$, the quantity η_n in Lemma 4.3 is, for all n large enough, at most

$$\Phi(n)\,4u_n\sqrt{\log n}\,\{3u_n/\sqrt{\log n}\}^{1/c_2}.$$

This is in turn at most

$$12\sqrt{18(\varkappa\vee 1)}\,\Phi(n)\sqrt{L(n)}\,\{\sqrt{\log\log n}\log^{-\beta/2} n\}$$

if we take $u_n = \log^\beta n$ for $\beta = 1/\{4(1+c_2)\}$.

Theorem 4.4 *Suppose that Assumptions* A1–A2 *and* (4.36) *hold, fix* $3\alpha = \beta = \frac{1}{4(c_2+1)}$, *and set* $u_n = \log^\beta n$, $\varphi_n = \log^\alpha n$. *Then, on* $B_1(2\log n, \varphi_n)$, *we have*

$$\sup_{T\ge 0}|A_n(T\wedge\tau_n) - A_n^*(T)| = \varepsilon(n)\left(\Phi(n)\sqrt{L(n)}\right),$$

where $\lim_{n\to\infty}\varepsilon(n) = 0$ *uniformly in* $c_2 < C < \infty$, *for each* $C > 0$. *Furthermore,*

$$\lim_{n\to\infty}\mathbb{P}[B_1(2\log n, \varphi_n)] \;=\; 0.$$

Proof. By the argument just completed, it is enough to examine the integrated difference

$$\int_0^{\psi_n \wedge T} \left| \Phi(ne^{-S_t}) - \Phi(ne^{-t}) \left\{ 1 - \frac{Z_t}{L(ne^{-t})} \right\} \right| \, \mathrm{d}t,$$

where $\psi_n = \log n - 2u_n\sqrt{\log n}$.

To this end, we use Lemma 3.2 with ne^{-t} for m and Z_t for s. On $B_1(2\log n, \varphi_n)$, and since, for $0 \le t \le \psi_n$, we have $ne^{-t} \ge \exp\{2u_n\sqrt{\log n}\}$, it follows that

$$\begin{aligned} |Z_t| &\le \varphi_n\sqrt{\log n} \;=\; \tfrac{1}{2}\log\{\exp(2\varphi_n\sqrt{\log n})\} \\ &\le \tfrac{1}{2}\log\{\exp(2u_n\sqrt{\log n})\} \;\le\; \tfrac{1}{2}\log\{ne^{-t}\}, \end{aligned}$$

so that the lemma can be applied. It then follows that

$$\begin{aligned} &|\Phi(ne^{-S_t}) - \Phi(ne^{-t})\exp\{-Z_t/L(ne^{-t})\}| \\ &\le \; \Phi(ne^{-t})\exp\{-Z_t/L(ne^{-t})\}X(n,t)\exp\{X(n,t)\}\,, \end{aligned} \tag{4.41}$$

where

$$X(n,t) \;=\; \frac{\varkappa Z_t^2}{L(ne^{-t})(\log n - t)}\,. \tag{4.42}$$

It is then also immediate from $e^{-x} - 1 + x < e^{|x|}x^2/2$ that

$$\begin{aligned} &\Phi(ne^{-t})|\exp\{-Z_t/L(ne^{-t})\} - 1 + Z_t/L(ne^{-t})| \\ &\le \; \tfrac{1}{2}\Phi(ne^{-t})\exp\{|Z_t|/L(ne^{-t})\}\{Z_t/L(ne^{-t})\}^2\,. \end{aligned} \tag{4.43}$$

Now, on $B_1(2\log n, \varphi_n)$, and for $0 \le t \le \psi_n$, we have

$$X(n,t) \;\le\; (6\varkappa/c_1)(\varphi_n/u_n)^2 \log\log n \;\le\; \lambda_1,$$

and also

$$|Z_t|/L(ne^{-t}) \;\le\; (6/c_1)(\varphi_n/u_n)\log\log n \;\le\; \lambda_2;$$

more precisely, in this range of t, by (4.36)

$$X(n,t) \;\le\; \frac{6\varkappa\varphi_n^2 \log n \log\log n}{c_1(\log n - t)^2} \quad \text{and} \quad \frac{|Z_t|}{L(ne^{-t})} \;\le\; \frac{6\varphi_n\sqrt{\log n}\log\log n}{c_1(\log n - t)}\,.$$

We also have the bound (4.39) for $\Phi(ne^{-t})$. Combining these, it follows that

$$\begin{aligned} &\left| \int_0^{\psi_n} \Phi(ne^{-S_t})\,\mathrm{d}t - \int_0^{\psi_n} \Phi(ne^{-t}) \left\{ 1 - \frac{Z_t}{L(ne^{-t})} \right\} \mathrm{d}t \right| \\ &\le \; \int_0^{\psi_n} \Phi(ne^{-t}) \left\{ e^{\lambda_1+\lambda_2}X(n,t) + \tfrac{1}{2}e^{\lambda_2}(Z_t/L(ne^{-t}))^2 \right\} \mathrm{d}t \\ &\le \; \frac{6}{c_1^2}(\varkappa c_1 e^{\lambda_1+\lambda_2} + 3e^{\lambda_2})\varphi_n^2\Phi(n)(\log\log n)^2 \int_{2u_n/\sqrt{\log n}}^1 u^{-2+1/c_2}\,\mathrm{d}u \qquad (4.44) \\ &\le \; \lambda\left(\Phi(n)\sqrt{L(n)}\right)(\log\log n)^{5/2}\varphi_n^2/u_n \\ &= \; \lambda\left(\Phi(n)\sqrt{L(n)}\right)\{(\log\log n)^{5/2}\log^{-\beta/3} n\}, \qquad (4.45) \end{aligned}$$

for some $\lambda > 0$. This completes the proof. Note that c_2 enters the bound implicitly, in the value of β, and hence in λ_1 and λ_2. $\square$

Remark. The restrictions imposed by (4.36) can be relaxed somewhat, to allow a little more freedom in both lower and upper bounds. For instance, the same proof can be used under the condition

$$(\log m)^{1-\gamma(c_2(m))} \;\leq\; L(m) \;\leq\; c_2(m)\log m \quad \text{for all} \quad m \geq m_0, \tag{4.46}$$

for any increasing function c_2 satisfying $c_2(m) = o(\log\log m/\log\log\log m)$, where $\gamma(c) := 1/\{32(c+1)\}$. Suitable choices of the parameters are now

$$\beta \;=\; \beta(n) \;:=\; 8\gamma(c_2(n)) \quad \text{and} \quad \alpha \;=\; \alpha(n) \;:=\; \beta(n)/3;$$

note that, with these definitions and with $\varphi_n = \log^{\alpha(n)} n$, we still have $\mathbb{P}[B_1^c(2\log n, \varphi_n)] \to 0$. This extra freedom enables the main transition, between the behaviour in the case of moderately growing $\Phi(n)$ and that when $\Phi(n)$ grows either faster or more slowly, to be understood in greater detail.

4.2 Fast growing Φ

We turn to the setting in which slowly varying $\Phi(n)$ grows faster than any power of $\log n$. In this case most of the random fluctuation in A_n takes place at times of order $L(n)$, where $L(n)$ goes to infinity (as required by A1) but slower than $\log n$. Our global condition determining this régime is

$$6L(n)\log L(n) \;\leq\; c_1 \log n, \tag{4.47}$$

where $c_1 = \{3(\varkappa \vee 1)\}^{-1}$ is as before. Note that, if $L(n) \leq \frac{c_1 \log n}{6\log\log n}$, then (4.47) is satisfied; the condition given in (4.36) was chosen to match neatly, though in view of the remark at the end of the previous section, this was not really necessary. Here, we first need a modification of Lemma 4.3, in order to be able to truncate the integrals defining $A_n(T \wedge \tau_n)$ and $A_n^*(T)$ as far as we need to.

Lemma 4.5 *Suppose that ψ_n is such that $6L(n)\log L(n) \leq \psi_n \leq c_1 \log n$, and that n is large enough to satisfy $L(n) \geq e^6$. Then, on the event $B_2(\psi_n)$, we have*

$$0 \;\leq\; \int_0^{\tau_n \wedge T} \Phi(ne^{-S_t})\,\mathrm{d}t - \int_0^{\psi_n \wedge T} \Phi(ne^{-S_t})\,\mathrm{d}t \;\leq\; 4\Phi(n)\{e^{-3} + c_1^{-1}\}; \tag{4.48}$$

$$\int_{\psi_n \wedge T}^{T \wedge \log n} \Phi(ne^{-t}) \left| 1 - \frac{Z_t}{L(ne^{-t})} \right| \mathrm{d}t \;\leq\; \Phi(n)\{2e^{-3} + \tfrac{3}{2}c_1^{-1} + 6e^{-9}2^k\}. \tag{4.49}$$

Proof. On $B_2(\psi_n)$, we have $\frac{2}{3}\log n \leq \tau_n \leq 2\log n$, implying immediately that $\psi_n \leq \min\{\tau_n, \log n\}$. Hence, if $T \leq \psi_n$, both of the quantities to be bounded in the lemma are zero. Note also, in preparation, that for $l \geq e^6$ and for any $x \geq 6l\log l$, we have

$$xe^{-x/4l} \;\leq\; 6l\log l \exp\{-3\log l/2\} \;=\; 6l^{-1/2}\log l \;\leq\; 2. \tag{4.50}$$

For the bound (4.48), since $S_t \geq t/2$ for $t \geq \psi_n$ on $B_2(\psi_n)$ and since $c_1 \leq 1/2(\varkappa \vee 1)$, we can apply Corollary 3.3 to give

$$\begin{aligned}\int_{\psi_n}^{c_1 \log n} \Phi(ne^{-S_t})\,\mathrm{d}t &\leq \Phi(n) \int_{\psi_n}^{c_1 \log n} e^{-t/4L(n)}\,\mathrm{d}t \\ &\leq 4L(n)\Phi(n)e^{-\psi_n/4L(n)} \leq 4L(n)^{-1/2}\Phi(n),\end{aligned}$$

by the definition of ψ_n. Then we also have

$$\int_{c_1 \log n}^{2 \log n} \Phi(ne^{-S_t})\,\mathrm{d}t \leq 2 \log n\, \Phi(n) \exp\{-c_1 \log n/4L(n)\} \leq 4c_1^{-1}\Phi(n),$$

this last by (4.50).

The argument for (4.49) is very similar. First, bounding $\int_{\psi_n}^{\log n} \Phi(ne^{-t})\,\mathrm{d}t$, it follows from Corollary 3.3 that

$$\int_{\psi_n}^{c_1 \log n} \Phi(ne^{-t})\,\mathrm{d}t \leq 2L(n)\Phi(n)e^{-\psi_n/4L(n)} \leq 2L(n)^{-1/2}\Phi(n),$$

and then that

$$\int_{c_1 \log n}^{\log n} \Phi(ne^{-t})\,\mathrm{d}t \leq \log n\, \Phi(n) \exp\{-c_1 \log n/2L(n)\} \leq c_1^{-1}\Phi(n).$$

For the remaining term, we first have

$$\begin{aligned}\int_{\psi_n}^{c_1 \log n} \frac{\Phi(ne^{-t})|Z_t|}{L(ne^{-t})}\,\mathrm{d}t &\leq \frac{\Phi(n)}{L(n^{1-c_1})} \int_{\psi_n}^{c_1 \log n} \tfrac{1}{2} t e^{-t/2L(n)}\,\mathrm{d}t \\ &\leq \frac{2L(n)^2\Phi(n)}{L(n^{1-c_1})} \int_{\psi_n/2L(n)}^{\infty} ue^{-u}\,\mathrm{d}u. \qquad (4.51)\end{aligned}$$

Now, for any $y \geq 1/2$,

$$\int_y^\infty ue^{-u}\,\mathrm{d}u = \int_0^\infty (y+v)e^{-y-v}\,\mathrm{d}v = e^{-y}(y+1) \leq 3ye^{-y},$$

so that (4.51) can be bounded, using (3.26) and (4.50), by

$$\frac{3L(n)\Phi(n)}{L(n^{1-c_1})}\,\psi_n e^{-\psi_n/2L(n)} \leq \frac{6L(n)\Phi(n)}{L(n^{1/2})}\,e^{-\psi_n/4L(n)} \leq \frac{6 \cdot 2^k \Phi(n)}{\{L(n)\}^{3/2}}.$$

Finally, using (5.54) and $c_1 < 1/2(\varkappa \vee 1)$, we have

$$\begin{aligned}\int_{c_1 \log n}^{\log n} \frac{\Phi(ne^{-S_t})|Z_t|}{L(ne^{-t})}\,\mathrm{d}t &\leq \tfrac{1}{2} \log n\, \Phi(ne^{-c_1 \log n}) \\ &\leq \tfrac{1}{2} \log n\, \Phi(n)\, e^{-c_1 \log n/2L(n)} \leq \tfrac{1}{2c_1}\Phi(n),\end{aligned}$$

again using (4.50). This completes the proof. □

Theorem 4.6 *Under Assumptions* A1–A2 *and* (4.47)*, set* $\psi_n = 6L(n)\log L(n)$ *and* $\varphi_n = L(n)^{1/6}$*. Then, on the event* $B_1(\psi_n, \varphi_n) \cap B_2(\psi_n)$*, and if* $L(n) \geq e^6$*, we have*

$$\sup_{T \geq 0} |A_n(T \wedge \tau_n) - A_n^*(T)| \;\leq\; \varepsilon(L(n)) \left(\Phi(n)\sqrt{L(n)}\right),$$

where $\lim_{m\to\infty} \varepsilon(m) = 0$*. Furthermore,*

$$\lim_{n\to\infty} \mathbb{P}[B_1^c(\psi_n, \varphi_n)] \;=\; \lim_{n\to\infty} \mathbb{P}[B_2^c(\psi_n)] \;=\; 0.$$

Proof. As before, on $B_2(\psi_n)$, we have $\frac{2}{3}\log n \leq \tau_n \leq 2\log n$, implying immediately that $\psi_n \leq \min\{\tau_n, \log n\}$. By Lemma 4.5, it is enough to bound the difference

$$\int_0^{\psi_n} \left| \Phi(ne^{-S_t}) - \Phi(ne^{-t}) \left\{ 1 - \frac{Z_t}{L(ne^{-t})} \right\} \right| \, dt.$$

By (3.26), we can use the inequality $L(ne^{-t}) \geq 2^{-k}L(n)$ for $0 \leq t \leq c_1 \log n$. Hence, on the event $B_1(\psi_n, \varphi_n)$, and noting that $\log n - \psi_n \geq \frac{1}{2}L(n)$ because of (4.47), we can bound the quantities $X(n,t)$ and $\{Z_t/L(ne^{-t})\}^2$ appearing in the proof of Theorem 4.4 by

$$2^{2k}\varphi_n^2\psi_n L(n)^{-2} \;=\; 6 \cdot 2^{2k} L(n)^{-2/3} \log L(n) \;\leq\; 36\, e^{-4}\, 2^{2k},$$

in the range $t \leq \psi_n$; thus they are both uniformly bounded in n, and asymptotically small as $n \to \infty$. Hence, using (4.41) and (4.43), it follows that

$$\begin{aligned} &\int_0^{\psi_n} \left| \Phi(ne^{-S_t}) - \Phi(ne^{-t}) \left\{ 1 - \frac{Z_t}{L(ne^{-t})} \right\} \right| \, dt \\ &\quad\leq\; \lambda L(n)^{-2/3} \log L(n) \int_0^{\psi_n} \Phi(ne^{-t}) \, dt \end{aligned}$$

for some $\lambda < \infty$. But now, from Corollary 3.3, it follows that

$$\begin{aligned} L(n)^{-2/3} \log L(n) \int_0^{\psi_n} \Phi(ne^{-t}) \, dt \;&\leq\; 2L(n)^{1/3}\Phi(n) \log L(n) \\ &=\; 2L(n)^{-1/6} \log L(n) \left(\Phi(n)\sqrt{L(n)}\right), \end{aligned}$$

proving the main assertion. The last statement follows from Corollary 4.2. □

5 The backward argument

We now turn to the case of functions $\Phi(n)$ that grow more slowly than any power of $\log n$. Here, the argument required and the approximations obtained are of rather different character to those of the previous section. In particular, we make use of properties of the Lévy process when looking backwards in time. Our setting is defined by requiring that $\lim_{n\to\infty} \Phi(n) = \infty$, but that L satisfies the following global condition:

$$L(m) = c_2(m) \log m, \quad \text{where} \quad \lim_{m\to\infty} c_2(m) \;=\; \infty. \tag{5.52}$$

To agree with A1, $c_2(m)$ must grow slowly enough, meaning that the integral in (3.24),

$$\int_2^\infty \frac{\mathrm{d}m}{c_2(m) m \log m}\,,$$

must diverge, a condition which excludes functions like $c_2(m) = \log^\varepsilon m$ for any $\varepsilon > 0$. One can think of $c_2(m) = \log\log m$ for $m \geq m_0$, as one possible example, in which case $\Phi(n) \asymp \log\log n$. Here, we no longer have Lemma 4.3 to help us. However, the argument of Theorem 4.4 is still good, if we restrict to taking the supremum over $0 \leq T \leq (1-\delta_n)\log n$, for some $\delta_n \to 0$ sufficiently slowly, and this gives us the following approximation of A_n by A_n^*.

Lemma 5.1 *Take $\alpha = 1/8$, $\varphi_n = \log^\alpha n$ and $\delta_n = \log^{-1/8} n$. Then, on the event $B_1(2\log n, \varphi_n)$, it follows that*

$$\sup_{0\leq T\leq (1-\delta_n)\log n} |A_n(T\wedge\tau_n) - A_n^*(T)| \;\leq\; \lambda\,\frac{1}{\log^{1/8} n}\,\frac{\Phi(n)\sqrt{\log n}}{c_2^*(n^{\delta_n})}\,,$$

for some $\lambda > 0$ and $c_2^*(m) = \inf_{r\geq m} c_2(r)$.

Proof. We argue as for Theorem 4.4, now with $\psi_n = (1-\delta_n)\log n$, noting that, for $t \leq \psi_n$,

$$|X(n,t)| \;\leq\; \frac{\varkappa\varphi_n^2 \log n}{(\log n - t)^2 c_2^*(n^{\delta_n})} \;\leq\; \frac{\varkappa}{\varphi_n^2 c_2^*(n^{\delta_n})}\,,$$

since $\delta_n > \varphi_n^2/\sqrt{\log n}$, and that

$$\frac{|Z_t|}{L(ne^{-t})} \;\leq\; \frac{\varphi_n\sqrt{\log n}}{(\log n - t)c_2^*(n^{\delta_n})} \;\leq\; \frac{1}{\varphi_n c_2^*(n^{\delta_n})}\,,$$

both of which are small in n. Then, arguing as for (4.44), and using the crude bound $\Phi(ne^{-t}) \leq \Phi(n)$, we have

$$\begin{aligned} |A_n(T\wedge\tau_n) - A_n^*(T)| \;&\leq\; \lambda\,\frac{\Phi(n)\varphi_n^2}{c_2^*(n^{\delta_n})}\int_{\delta_n}^1 u^{-2}\,\mathrm{d}u \\ &\leq\; \lambda\,\frac{\Phi(n)\varphi_n^2}{\delta_n c_2^*(n^{\delta_n})} \;\leq\; \lambda\,\frac{1}{\log^{1/8} n}\,\frac{\Phi(n)\sqrt{\log n}}{c_2^*(n^{\delta_n})}\,, \end{aligned}$$

for $T \leq (1-\delta_n)\log n$, as required. □

To see that differences of this order are relatively small, we now make some variance calculations, for which we introduce the notation

$$-W(v) \;:=\; \int_0^{v\log n} g(n,t)(S_t - t)\,\mathrm{d}t, \tag{5.53}$$

where

$$g(n,t) \;:=\; \Phi(ne^{-t})/L(ne^{-t}) \;=\; ne^{-t}\Phi'(ne^{-t}) \;=\; -\frac{\mathrm{d}}{\mathrm{d}t}\left\{\Phi(ne^{-t})\right\}. \tag{5.54}$$

It thus follows that

$$W(v) = A_n^*(v\log n) - \mathbb{E}\{A_n^*(v\log n)\} \tag{5.55}$$

whenever $vL(n) \leq \log n$.

Lemma 5.2 *For $0 \le v \le 1$ and n large enough, we have*

$$\mathrm{Var}\, A_n^*(v \log n) \;\ge\; \lambda(v \wedge \tfrac{1}{2})^3 \Phi(n)^2 \log n / c_2(n)^2;$$

for $0 < \delta < 1$ and for $0 \le v \le (1-\delta)$, we have

$$\mathrm{Var}\, A_n^*(v \log n) \;\le\; \lambda \Phi(n)^2 \log n \, \frac{\log(1/\delta)}{c_2^*(n^\delta)} \,,$$

where $c_2^*(m) = \inf_{r \ge m} c_2(r)$.

Proof. From Lemma 3.2, it follows that, for $0 \le v \le \frac{1}{2}$ and $t = v \log n$,

$$\Phi(ne^{-t}) \ge \Phi(n)\, e^{-t/L(n)} \exp\{-\varkappa v^2/c_2(n)\} \,,$$

and, from (3.26), that $L(n)/L(ne^{-t}) \ge 2^{-k}$, implying that, for n so large that $\varkappa/\{4c_2(n)\} \le 1$, we have

$$g(n,t) \ge 2^{-k} e^{-1} \Phi(n)\, e^{-t/L(n)}/L(n) \,,$$

where $g(n,t)$ is as in (5.54). Now $Z_t = S_t - t$ has independent increments with zero means, and $\mathrm{Var}\, Z_t = \sigma^2 t$. Hence, for any $0 \le v \le 1/2$, recalling (5.55), we have

$$\begin{aligned}
\mathrm{Var}\, A_n^*(v \log n) &= 2 \int_0^{v \log n} \int_0^t g(n,t) g(n,u) \sigma^2 u \,\mathrm{d}u\,\mathrm{d}t \\
&\ge 2^{1-2k} e^{-2} \Phi(n)^2 \int_0^{v/c_2(n)} \int_0^w e^{-w-z} \sigma^2 z L(n) \,\mathrm{d}z\,\mathrm{d}w \\
&\ge 2^{1-2k} e^{-2} \sigma^2 \Phi(n)^2 c_2(n) \log n \cdot \tfrac{1}{6} (v/c_2(n))^3 e^{-1/c_2(n)} \\
&\ge 2^{1-2k} \sigma^2 \Phi(n)^2 c_2(n) \log n \cdot \tfrac{1}{6} (v/c_2(n))^3 e^{-3} \,,
\end{aligned}$$

for all n large enough. This proves the first inequality, since this lower bound with $v = 1/2$ is a lower bound for larger v also.

For the second part, we recall (4.35):

$$\mathrm{Var}\, A_n^*(v \log n) \;=\; \sigma^2 \int_0^{v \log n} \{\Phi(ne^{-t}) - \Phi(n^{1-v})\}^2 \,\mathrm{d}t,$$

whenever $0 \le v \le 1$. Now, from the representation (3.24), it follows that, for $0 < T \le (1-\delta)\log n$,

$$\begin{aligned}
&\int_0^T \{\Phi(ne^{-t}) - \Phi(ne^{-T})\}^2 \sigma^2 \,\mathrm{d}t \\
&\quad\le \int_0^T \Phi^2(ne^{-t}) \left\{ 1 - \exp\left(-\frac{1}{c_2^*(n^\delta)} \int_{ne^{-T}}^{ne^{-t}} \frac{\mathrm{d}y}{y \log y} \right) \right\}^2 \mathrm{d}t \\
&\quad= \int_0^T \Phi^2(ne^{-t}) \left\{ 1 - \left(\frac{1 - T/\log n}{1 - t/\log n} \right)^{1/c_2^*(n^\delta)} \right\}^2 \mathrm{d}t \\
&\quad\le T\, \Phi^2(n) \{1 - \delta^{1/c_2^*(n^\delta)}\} \,,
\end{aligned}$$

and the second part is proved. □

In particular, the lower bound shows that the standard deviation of $A_n^*(T)$ is at least as big as a constant times $\Phi(n)\sqrt{\log n}/c_2(n)$ for $T \geq \frac{1}{2}\log n$. By comparison, the differences in Lemma 5.1 are typically much smaller, because of the factor $\log^{-1/8} n$; recall that $c_2(n)$ grows rather slowly with n, and certainly not as fast as a power of $\log n$.

Note also that, if $\delta = \delta_n \to 0$ sufficiently slowly, the upper bound can be made to grow more slowly that $\Phi^2(n)\log n$. For example, with $c_2(m) = \log\log m$ and therefore $\Phi(n) \asymp \log\log n$, one could take $\delta_m = 1/\log\log m$, giving an upper bound of order

$$O(\log n \log\log n \log\log\log n) = o(\log n\{\log\log n\}^2)\,.$$

In general, taking $\delta = \delta_n$ to be the solution of the equation $\log(1/\delta) = \sqrt{c_2^*(n^\delta)}$ gives both $\delta_n \to 0$ and $\mathrm{Var}\, A_n((1-\delta_n)\log n) = o(\Phi^2(n)\log n)$. Thus, *almost* up to the time $\log n$, the compensator A_n behaves very much like the simpler integral process A_n^*, but the common scale of their fluctuations is of smaller order than that of $A_n(\infty)$, which, by Corollary 3.4, has variance of order $\Psi_2(n) \asymp \Phi^2(n)\log n$.

We now turn to approximating $A_n(\infty)$. As before, it is enough to consider $A_n(\tau_n)$, which we can write in the form

$$A_n(\tau_n) \;=\; \int_0^{\tau_n} \Phi(ne^{-S_t})\,\mathrm{d}t \;=\; \int_0^{\tau_n} \Phi(ne^{-S_{\tau_n - v}})\,\mathrm{d}v. \tag{5.56}$$

We now define the process $\widehat{Z}_n$ by the equation

$$\widehat{Z}_n(v) \;:=\begin{cases} S_{\tau_n -} - v - S_{(\tau_n - v)-} & \text{for } v < \tau_n\,, \\ S_{\tau_n -} - \tau_n & \text{for } v \geq \tau_n\,, \end{cases} \tag{5.57}$$

and we look for a suitable approximation to $A_n(\tau_n)$ when the paths of $\widehat{Z}_n$ are 'nice'.

Very much as before, we define good events, for $\varphi, \psi > 0$,

$$\widehat{B}_1(\varphi, n) \;:=\; \left\{ \sup_{0 \leq v \leq 2\log n} (v \vee 1)^{-1/2} |\widehat{Z}_n(v)| \leq \varphi \right\}; \tag{5.58}$$

$$\widehat{B}_2(\psi, n) \;:=\; \{\log n - S_{\tau_n -} \leq \psi\}; \tag{5.59}$$

$$\widehat{B}_3(T, \psi, n) \;:=\; \left\{ \int_0^T v^{-1} |\widehat{Z}_n(v)|\,\mathrm{d}v \leq \psi \right\}, \tag{5.60}$$

whose probabilities we wish to show are large. The next two lemmas make this precise; we recall the definition (4.29) of the event $B_0(n)$.

Lemma 5.3 *For any $T, \varphi, \psi > 0$, we have*

$$\begin{aligned} \mathbb{P}[\widehat{B}_1^c(\varphi, n) \cap B_0(n)] &\leq \sigma^2\varphi^{-2}(72 + 29\log\log n); \\ \mathbb{P}[\widehat{B}_3^c(T, \psi, n) \cap B_0(n)] &\leq 6\sigma\psi^{-1}\sqrt{T}. \end{aligned}$$

Proof. In order to make the calculations, it is convenient to exploit the explicit Itô construction of the process S [3, Proposition 1.3]. For H a Poisson point process on $\mathbb{R}_+^2$ with intensity measure $\mathrm{d}t\,\nu_0(\mathrm{d}x)$ we can define

$$S_t \;:=\; \int_{]0,t]\times\mathbb{R}_+} x\,H(\mathrm{d}t\,\mathrm{d}x); \qquad S_t^{(2)} \;:=\; \int_{]0,t]\times\mathbb{R}_+} x^2\,H(\mathrm{d}t\,\mathrm{d}x),$$

S being a copy of our original subordinator. We also define the family of random point measures μ_t on $\mathbb{R}_+$ by

$$\mu_t[0,x] \;:=\; H(]0,t[\,\times[0,x]).$$

We then define the family of σ-fields

$$\mathcal{F}_{-t} \;:=\; \sigma\{\mu_t,\, H|_{[t,\infty]\times\mathbb{R}_+}\}, \quad t \ge 0,$$

so that $\mathcal{F}_s \subset \mathcal{F}_{s'}$ whenever $s \le s' \le 0$. Then direct calculations show that the processes $(M^{(l)}(t),\, t>0)$, $l=1,2,3$, are reversed martingales with respect to the filtration $\{\mathcal{F}_s,\, s<0\}$, with means 1, σ^2 and zero, respectively, where

$$M^{(1)}(t) := t^{-1}S_{t-}, \quad M^{(2)}(t) := t^{-1}S_{t-}^{(2)} \quad \text{and} \quad M^{(3)}(t) \;:=\; (t^{-1}S_{t-}-1)^2 - t^{-2}S_{t-}^{(2)}.$$

Thus it is immediate from the optional sampling theorem that

$$\mathbb{E}\{\tau_n^{-1}S_{\tau_n-}\} \;\le\; 1; \qquad \mathbb{E}\{\tau_n^{-1}S_{\tau_n-}^{(2)}\} \;\le\; \sigma^2. \tag{5.61}$$

It also follows that $\mathbb{E}M^{(3)}(t\vee\tau_n)=0$ for any $t>0$, which, taking $t=\frac{1}{2}\log n$, implies that

$$\mathbb{E}\left\{(\tau_n^{-1}S_{\tau_n-}-1)^2\mathbf{1}\{\tau_n \ge \tfrac{1}{2}\log n\}\right\} \le 2\sigma^2/\log n. \tag{5.62}$$

Furthermore, for $v<\tau_n$, the equality $\mathbb{E}\{M(\tau_n-v)\,|\,\mathcal{F}_{-\tau_n}\} = W(\tau_n)$ a.s. also implies that, for such v,

$$\mathbb{E}\{U_n(v)^2\,|\,\mathcal{F}_{-\tau_n}\} \;=\; v\,\frac{\tau_n^{-1}S_{\tau_n-}^{(2)}}{\tau_n(\tau_n-v)}\,, \tag{5.63}$$

where

$$U_n(v) \;:=\; (\tau_n-v)^{-1}S_{(\tau_n-v)-} - \tau_n^{-1}S_{\tau_n-}\,.$$

We thus have the expression

$$\widehat{Z}_n(v) \;=\; (v\wedge\tau_n)\{\tau_n^{-1}S_{\tau_n-}-1\} - (\tau_n-v)_+U_n(v), \quad v\ge 0, \tag{5.64}$$

as an alternative representation for $\widehat{Z}_n$, in addition to (5.57). Taking expectations conditional of $\mathcal{F}_{-\tau_n}$, we thus obtain

$$\begin{aligned}\mathbb{E}\{|\widehat{Z}_n(v)|\,|\,\mathcal{F}_{-\tau_n}\} \;&\le\; (v\wedge\tau_n)|\tau_n^{-1}S_{\tau_n-}-1| + (\tau_n-v)_+\mathbb{E}\{|U_n(v)|\,|\,\mathcal{F}_{-\tau_n}\}\\ &\le\; v|\tau_n^{-1}S_{\tau_n-}-1| + \sqrt{v}\,\sqrt{\tau_n^{-1}S_{\tau_n-}^{(2)}},\end{aligned}$$

the last inequality from (5.63). Multiplying by $\mathbf{1}\{\tau_n \ge \frac{1}{2}\log n\}$ and taking expectations thus yields

$$\mathbb{E}(|\widehat{Z}_n(v)|\mathbf{1}\{\tau_n \ge \tfrac{1}{2}\log n\}) \;\le\; v\sigma\sqrt{2/\log n} + \sigma\sqrt{v} \;\le\; 3\sigma\sqrt{v}, \tag{5.65}$$

for $0 \le v \le 2\log n$, in view of (5.61) and (5.62). The second inequality now follows from Markov's inequality, because

$$\mathbb{E}\left\{\int_0^T v^{-1}|\widehat{Z}_n(v)|\,dv\mathbf{1}\{B_0(n)\}\right\} \;\le\; 3\sigma\int_0^T v^{-1/2}\,dv.$$

It also follows from (5.64) that, for any $\varphi > 0$ and for $v < \tau_n$,

$$\{|\widehat{Z}_n(v)| > \varphi\sqrt{v \vee 1}\} \subset \{|\tau_n^{-1}S_{\tau_n-} - 1| > \tfrac{1}{2}\varphi v^{-1/2}\} \cup \{(\tau_n - v)|U_n(v)| > \tfrac{1}{2}\varphi v^{1/2}\}\,.$$

The first event happens for some $v < \tau_n$ only if $|\tau_n^{-1}S_{\tau_n-} - 1| > \frac{1}{2}\varphi\tau_n^{-1/2}$, and the probability of this happening on the event $B_0(n)$ is at most

$$\begin{aligned}
&\mathbb{P}\Big[|\tau_n^{-1}S_{\tau_n-} - 1|\mathbf{1}\{\tau_n \ge \tfrac{1}{2}\log n\} > \tfrac{1}{2}\varphi(2\log n)^{-1/2}\Big] \\
&\quad\le\; \frac{2\sigma^2}{\log n}\cdot\frac{2\log n}{\varphi^2} \;=\; 4\sigma^2\varphi^{-2},
\end{aligned}$$

by (5.62). For the second, using Kolmogorov's inequality much as in the proof of Lemma 4.1, for $r \ge 1$ such that $2^r \le \frac{1}{2}\tau_n$, we have

$$\begin{aligned}
&\mathbb{P}\Big[\sup_{2^{r-1}\le v\le 2^r} v^{-1/2}(\tau_n - v)|U_n(v)| > \tfrac{1}{2}\varphi \,|\, \mathcal{F}_{-\tau_n}\Big] \\
&\quad\le\; \mathbb{P}\Big[\sup_{0\le v\le 2^r}|U_n(v)| > \tfrac{1}{2}\tau_n^{-1}\varphi 2^{(r-1)/2} \,|\, \mathcal{F}_{-\tau_n}\Big] \\
&\quad\le\; \frac{4\mathbb{E}\{|U_n(2^r)|^2 \,|\, \mathcal{F}_{-\tau_n}\}\tau_n^2}{\varphi^2\, 2^{r-1}} \\
&\quad\le\; 16\varphi^{-2}\tau_n^{-1}S^{(2)}_{\tau_n-},
\end{aligned}$$

using (5.63). Adding over all such r, and including the v-intervals $]0,1[$ and $]2^r, \frac{1}{2}\tau_n]$, it follows that

$$\mathbb{P}\Big[\sup_{0\le v\le\tau_n/2} v^{-1/2}(\tau_n - v)|U_n(v)|\mathbf{1}\{B_0(n)\} > \tfrac{1}{2}\varphi\Big\} \;\le\; 20\sigma^2\varphi^{-2}(1 + \lceil\log_2\log n\rceil), \tag{5.66}$$

from (5.61). For $\frac{1}{2}\log n < v < \tau_n$, we use (5.57) to give

$$\widehat{Z}_n(v) \;=\; Z_{\tau_n-} - Z_{(\tau_n-v)-},$$

so that

$$B_0(n) \cap \left\{\sup_{\frac{1}{2}\tau_n\le v\le\tau_n} v^{-1/2}|\widehat{Z}_n(v)| > \varphi\right\} \;\subset\; \left\{\sup_{0\le u\le 2\log n}|Z_u| > \tfrac{1}{4}\varphi\sqrt{\log n}\right\},$$

the latter event, by Kolmogorov's inequality, having probability at most $32\sigma^2\varphi^{-2}$. Finally, again by Kolmogorov's inequality,

$$\mathbb{P}[B_0(n)^c] \;\le\; \mathbb{P}\Big[\sup_{0\le u\le 2\log n} |Z_u| > \tfrac{1}{2}\log n\Big] \;\le\; 8\sigma^2/\log n.$$

From these last two bounds and from (5.66), the lemma follows. □

Lemma 5.4 *If $\psi_n \to \infty$, then* $\lim_{n\to\infty} \mathbb{P}[\widehat{B}_2^c(\psi_n, n)] = 0$.

Proof. Simply note that $\widehat{B}_2^c(x,n) \subset \widehat{B}_2^c(y,n)$ whenever $x > y$, so that then

$$\pi_n(x) \;:=\; \mathbb{P}[\widehat{B}_2^c(x,n)] \;\le\; \pi_n(y),$$

and that

$$\lim_{n\to\infty} \pi_n(x) \;=\; \frac{\int_x^\infty N_0(u)\,\mathrm{d}u}{\int_0^\infty N_0(u)\,\mathrm{d}u} \;=:\; \pi(x),$$

by the renewal theorem [2, p. 99], with $\lim_{x\to\infty}\pi(x) = 0$. Hence, given $\varepsilon > 0$, pick x so that $\pi(x) < \varepsilon/2$, and then n_x such that $\pi_n(x) \le \varepsilon$ and $\psi_n \ge x$ for all $n \ge n_x$; it then follows that $\pi_n(\psi_n) \le \pi_n(x) \le \varepsilon$ for all $n \ge n_x$. □

With these preparations, we are now in a position to approximate the behaviour of $A_n(\tau_n)$, and indeed of the whole process $A_n(t\wedge\tau_n)$.

Theorem 5.5 *Suppose that Assumptions* A1–A2 *and* (5.52) *hold. Fix $\alpha = 1/8$, $\beta = 1/4$, and set $v_n := 4\log^{2\alpha} n$. Then, on the event*

$$B_0(n) \cap \widehat{B}_1(\log^\alpha n, n) \cap \widehat{B}_2(\log^\beta n, n) \cap \widehat{B}_3(2\log n, \sqrt{c_2^*(e^{v_n})\log n}, n),$$

it follows that

$$\Big(\Phi(n)\sqrt{\log n}\Big)^{-1} \sup_{t\ge 0} \left| A_n(t\wedge\tau_n) - \int_{(\tau_n - t)_+}^{\tau_n} \Phi(e^v)\,\mathrm{d}v \right| \;\to\; 0.$$

Furthermore,

$$\lim_{n\to\infty} \mathbb{P}[B_0(n) \cap \widehat{B}_1(\log^\alpha n, n) \cap \widehat{B}_2(\log^\beta n, n) \cap \widehat{B}_3(2\log n, \sqrt{c_2^*(e^{v_n})\log n}, n)] = 1.$$

Here, $c_2^*(m)$ is defined as in Lemma 5.1.

Proof. Recalling (5.56), we can write

$$\begin{aligned}
A_n(t\wedge\tau_n) &= \int_0^{(t\wedge\tau_n)} \Phi(ne^{-S_u})\,\mathrm{d}u \\
&= \int_{(\tau_n-t)_+}^{\tau_n} \Phi(e^v)\,\mathrm{d}v + \int_{(\tau_n-t)_+}^{\tau_n} \big\{\Phi(e^{v+D_n}) - \Phi(e^v)\big\}\,\mathrm{d}v \\
&\quad + \int_{(\tau_n-t)_+}^{\tau_n} \Big\{\Phi(e^{v+D_n+\widehat{Z}_n(v)}) - \Phi(e^{v+D_n})\Big\}\,\mathrm{d}v, \qquad (5.67)
\end{aligned}$$

where $D_n := \log n - S_{\tau_n -} \geq 0$. The second of the integrals in (5.67) is nonnegative, and no larger than

$$\int_0^{\tau_n} \left\{ \Phi(e^{v+D_n}) - \Phi(e^v) \right\} \mathrm{d}v \;\leq\; \int_{\tau_n}^{\tau_n + D_n} \Phi(e^v)\, \mathrm{d}v \;\leq\; \Phi(n^3) \log^\beta n, \tag{5.68}$$

on the event $B_0(n) \cap \widehat{B}_2(\log^\beta n, n)$. Note also that, for any $r \geq 1$ and n such that $c_2^*(n) \geq 1$,

$$1 \;\leq\; \frac{\Phi(n^r)}{\Phi(n)} \;=\; \exp\left\{ \int_n^{n^r} \frac{\mathrm{d}y}{yL(y)} \right\} \;\leq\; \exp\left\{ (\log\log(n^r) - \log\log n)/c_2^*(n) \right\} \;=\; r^{1/c_2^*(n)} \;\leq\; r; \tag{5.69}$$

hence, from (5.68), the second of the integrals in (5.67) is of smaller order than $\Phi(n)\sqrt{\log n}$ on the event $B_0(n) \cap \widehat{B}_2(\log^\beta n, n)$.

To control the third of the integrals in (5.67), we bound

$$\int_0^{\tau_n} \left| \Phi(e^{v+D_n+\widehat{Z}_n(v)}) - \Phi(e^{v+D_n}) \right| \mathrm{d}v\,. \tag{5.70}$$

On $\widehat{B}_1(\log^\alpha n, n)$, we have $v^{-1}|\widehat{Z}_n(v)| \leq 1/2$ if $v \geq v_n$. So split the range of the integral into $0 < v \leq v_n$ and $v_n \leq v \leq \tau_n$. In the lower range, on $\widehat{B}_1(\log^\alpha n, n) \cap \widehat{B}_2(\log^\beta n, n)$, the exponents $v + D_n$ and $v + D_n + \widehat{Z}_n(v)$ are bounded above by

$$v_n + \log^\beta n + \log^\alpha n \sqrt{v_n} \;\leq\; 7 \log n,$$

implying, together with (5.69), that (5.70) is bounded above by $7\Phi(n)v_n$ for all n large enough, and this is $o(\Phi(n)\sqrt{\log n})$ by choice of α. In the upper range, we can apply Lemma 3.2 to $\Phi(e^{v+D_n+\widehat{Z}_n(v)})$, very much as in the proof of Theorem 4.4, because here $|\widehat{Z}_n(v)| \leq \frac{1}{2}(v+D_n)$. The quantity $\widehat{X}(n,v)$, analogous to $X(n,t)$ of (4.42), is bounded for $v \geq v_n$ by

$$\widehat{X}(n,v) \;\leq\; \frac{\varkappa|\widehat{Z}_n(v)|^2}{c_2(e^{v+D_n})v(v+D_n)} \;\leq\; \frac{\varkappa \log^{2\alpha} n}{v c_2^*(e^{v_n})} \;\leq\; \frac{\varkappa}{c_2^*(e^{v_n})},$$

and

$$\frac{|\widehat{Z}_n(v)|}{L(e^{v+D_n})} \;\leq\; \frac{\log^\alpha n}{c_2^*(e^{v_n})\sqrt{v}} \;\leq\; \frac{1}{2c_2^*(e^{v_n})},$$

giving

$$\int_{v_n}^{\tau_n} \left| \Phi(e^{v+D_n+\widehat{Z}_n(v)}) - \Phi(e^{v+D_n}) \right| \mathrm{d}v \;\leq\; \int_{v_n}^{\tau_n} \Phi(e^{v+D_n}) \frac{|\widehat{Z}_n(v)|}{v c_2^*(e^v)}\, \mathrm{d}v + \lambda \left(\frac{\Phi(e^{\tau_n+D_n}) \log^{2\alpha} n}{c_2^*(e^{v_n})} \int_{v_n}^{\tau_n} v^{-1}\, \mathrm{d}v \right), \tag{5.71}$$

for some positive constant $\lambda < \infty$. On $B_0(n) \cap \widehat{B}_2(\log^\beta n, n)$, and from (5.69), we have $\Phi(e^{\tau_n+D_n}) \leq 3\Phi(n)$ for all n large enough, so that the second term in (5.71) is of order $o(\Phi(n)\sqrt{\log n})$. The first term is bounded on

$$B_0(n) \cap \widehat{B}_2(\log^\beta n, n) \cap \widehat{B}_3(2\log n, \sqrt{c_2^*(e^{v_n}) \log n}, n)$$

by

$$\Phi(n^3)\sqrt{\log n}\,/\sqrt{c_2^*(e^{v_n})} \;=\; o(\Phi(n)\sqrt{\log n}),$$

again by (5.69). This completes the proof of the main statement. The final assertion follows from Lemmas 4.1, 5.3 and 5.4. □

6 Approximation theorems

We can now build on the results of the previous sections to derive central limit approximations for $\mathcal{K}_n$. The starting point is the functional central limit theorem for the Lévy process itself. Defining the process W_m by $W_m(t) := \sigma^{-1}m^{-1/2}Z_{mt}$, it follows that

$$W_m \;\to_d\; W \quad \text{in } D_1[0,\infty[\quad \text{as } m\to\infty, \tag{6.72}$$

where W is standard Brownian motion and $D_1[0,\infty[$ denotes the space of càdlàg functions $x\colon [0,\infty[\,\to\mathbb{R}$ satisfying $\lim_{t\to\infty} t^{-1}x(t) = 0$, endowed with the metric $\rho_1(x,y) := \sup_{t\ge 0}(t\vee 1)^{-1}|x(t)-y(t)|$ (Müller [13], Satz 1). As a consequence of the central limit theorem for the renewal processes [6, Section XI.5], it also follows that

$$U_n \;:=\; (\tau_n - \log n)/\{\sigma\sqrt{\log n}\} \;\to_d\; \mathcal{N}(0,1) \quad \text{as } n\to\infty. \tag{6.73}$$

We shall also be interested in approximations which are not given in the form of limit theorems, but are instead expressed in terms of bounds on a distance between the distributions of the processes considered, taken here to be the appropriate bounded Wasserstein distances. For probability measures Q and Q' on a metric space $(\mathcal{X},\rho)$, the bounded Wasserstein distance $d_{BW}(Q,Q')$ is defined to be $\sup_{f\in\mathcal{W}}|\int f\,\mathrm{d}Q - \int f\,\mathrm{d}Q'|$, where $\mathcal{W}$ denotes the bounded Lipschitz functions on $\mathcal{X}$:

$$\mathcal{W} \;:=\; \mathcal{W}_{\mathcal{X},\rho} \;:=\; \{f\colon \mathcal{X}\to\mathbb{R} : \|f\|\le 1,\, L(f)\le 1\},$$

and $L(f) := \sup_{x\ne x'\in\mathcal{X}}|f(x)-f(x')|/\rho(x,x')$. The distance d_{BW} metrises weak convergence in $(\mathcal{X},\rho)$ (Dudley [5], Theorem 8.3). Note also that if, for each $n\ge 1$, the random elements X_n and Y_n of $(\mathcal{X},\rho)$ are on the same probability space, then $d_{BW}(\mathcal{L}(X_n),\mathcal{L}(Y_n))\to 0$ if, for each $\varepsilon>0$, $\mathbb{P}[\rho(X_n,Y_n)>\varepsilon]\to 0$. If $\mathcal{X}$ is the space $D_1[0,\infty[$ defined above, we shall refer to $\mathcal{W}^1$ and d_{BW}^1; if $\mathcal{X}$ is the space $D_0[0,\infty[$ of càdlàg functions $x\colon[0,\infty[\,\to\mathbb{R}$ having finite limits as $t\to\infty$, endowed with the metric $\rho_0(x,y) := \sup_{t\ge0}|x(t)-y(t)|$, we shall refer to $\mathcal{W}^0$ and d_{BW}^0, and, if $\mathcal{X}=\mathbb{R}$, we shall write $d_{BW}^{\mathbb{R}}$.

6.1 Moderate growth

We begin with a setting of moderate growth, in which $L(n)\asymp\log n$, so that (4.36) is in force. In order to describe the behaviour of $\mathcal{K}_n$, we first define a centred and normalized version $\mathcal{K}_n^{(1)}$ of the process by

$$\mathcal{K}_n^{(1)}(u) \;:=\; \left(\Phi(n)\sqrt{\log n}\right)^{-1}\left\{\mathcal{K}_n(u\log n) - \log n\int_0^{(u\wedge 1)}\Phi(n^{1-v})\,\mathrm{d}v\right\},$$

whose distribution we approximate by that of $Y_n^{(1)}$, where

$$Y_n^{(1)}(u) \;:=\; \sigma \int_0^{(u\wedge 1)} h_n^{(1)}(v)\, W(v)\, \mathrm{d}v,$$

with

$$h_n^{(1)}(u) \;:=\; \frac{\Phi(n^{1-u})\log n}{\Phi(n)L(n^{1-u})}.$$

Note that $h_n^{(1)}(u) \geq 0$ for all u, and that, from (5.54),

$$\int_0^1 h_n^{(1)}(u)\, \mathrm{d}u \;=\; 1 - \Phi(1)/\Phi(n) \;\leq\; 1 \qquad (n \geq 1).$$

Theorem 6.1 *If Assumptions* A1–A2 *hold, and* $L(n) \asymp \log n$, *then*

$$d_{BW}(\mathcal{L}(\mathcal{K}_n^{(1)}), \mathcal{L}(Y_n^{(1)})) \;\to\; 0 \quad \textit{as } n \to \infty\,.$$

Proof. We begin by writing

$$\mathcal{K}_n^{(1)}(u) - \widetilde{Y}_n^{(1)}(u) \;=\; r_n\Big\{(\mathcal{K}_n(u\log n) - A_n(u\log n))$$
$$+ (A_n(u\log n) - A_n(\{u\log n\}\wedge\tau_n)) + (A_n(\{u\log n\}\wedge\tau_n) - A_n^*(u\log n))\Big\},$$

where $r_n := \big(\Phi(n)\sqrt{\log n}\big)^{-1}$ and

$$\widetilde{Y}_n^{(1)} \;:=\; A_n^*(u\log n) - \log n \int_0^{u\wedge 1} \Phi(n^{1-v})\, \mathrm{d}v$$
$$=\; -\sigma \int_0^{(u\wedge 1)} h_n^{(1)}(v)\, W_{\log n}(v)\, \mathrm{d}v.$$

Now we have $r_n \sup_{u\geq 0}|\mathcal{K}_n(u\log n) - A_n(u\log n)| \to_p 0$ by Lemma 2.6 and Corollary 3.4, then $r_n \sup_{u\geq 0}|A_n(u\log n) - A_n(\{u\log n\}\wedge\tau_n)| \to_p 0$ by Lemma 2.2, and finally, by Theorem 4.4, $r_n \sup_{u\geq 0}|A_n(\{u\log n\}\wedge\tau_n) - A_n^*(u\log n)| \to_p 0$. Hence it follows that

$$d_{BW}^0(\mathcal{L}(\mathcal{K}_n^{(1)}), \mathcal{L}(\widetilde{Y}_n^{(1)})) \;\to\; 0 \quad \text{as } n \to \infty.$$

To conclude the proof, we now need to show that $\sup_{f\in\mathcal{W}^0}|\mathbb{E}f(\widetilde{Y}_n^{(1)}) - \mathbb{E}f(Y_n^{(1)})| \to 0$ as $n \to \infty$. To do so, for any $f \in \mathcal{W}^0$, define $f_n\colon D_1[0,\infty[\, \to \mathbb{R}$ by $f_n(w) := f(H_n(w))$, where $H_n(w)(u) := \int_0^{(u\wedge 1)} h_n^{(1)}(v)w(v)\,\mathrm{d}v$. Note that, for $w, w' \in D_1[0,\infty[$ and any $u \geq 0$,

$$|H_n(w)(u) - H_n(w')(u)| \;=\; \left|\int_0^{(u\wedge 1)} h_n^{(1)}(v)(w(v) - w'(v))\,\mathrm{d}v\right|$$
$$\leq\; \rho_1(w,w')\int_0^1 h_n^{(1)}(v)\,\mathrm{d}v \;\leq\; \rho_1(w,w').$$

Hence, for any $f \in \mathcal{W}^0$, it follows that $f_n \in \mathcal{W}^1$, and hence that

$$|\mathbb{E}f(\widetilde{Y}_n^{(1)}) - \mathbb{E}f(Y_n^{(1)})| \;=\; |\mathbb{E}f_n(W_{\log n}) - \mathbb{E}f_n(W)| \;\leq\; d_{BW}^1(\mathcal{L}(W_{\log n}), \mathcal{L}(W)).$$

The theorem now follows from (6.72). □

Theorem 6.2 *Under the assumptions of* Theorem 6.1, *if in addition $L(n) \sim \gamma \log n$ for some $0 < \gamma < \infty$, then*

$$\mathcal{K}_n^{(1)} \ \to_d \ Y^{(1)} \quad \textit{in } D_0[0,\infty[\quad \textit{as } n \to \infty,$$

where

$$Y^{(1)}(u) \ := \ \sigma \int_0^{(u\wedge 1)} \gamma^{-1}(1-v)^{(1-\gamma)/\gamma} W(v)\, \mathrm{d}v.$$

Proof. If $L(n) \sim \gamma \log n$, then $h_n^{(1)}(u) \to \gamma^{-1}(1-u)^{(1-\gamma)/\gamma}$ uniformly in $0 \le u \le 1-\delta$, for any $\delta > 0$; furthermore,

$$\limsup_{n\to\infty} \int_{1-\delta}^1 h_n^{(1)}(v)\, \mathrm{d}v \ = \ \limsup_{n\to\infty} \{\Phi(n^\delta) - \Phi(1)\}/\Phi(n) \ \le \ \delta^{\gamma'}$$

for any $\gamma' < \gamma$, and $\int_{1-\delta}^1 h^{(1)}(v)\, \mathrm{d}v = \delta^\gamma$. Hence

$$\mathbb{E}\left\{\sup_{u\ge 0} |Y_n^{(1)}(u) - Y^{(1)}(u)|\right\} \le \sqrt{\frac{2}{\pi}} \int_0^1 |h_n^{(1)}(v) - h(v)|\, \mathrm{d}v \ \to \ 0,$$

proving the theorem. □

Examples. Suppose, for some $0 < \gamma < \infty$, that $\widetilde{S}$ is a subordinator such that $\widetilde{L}(n) \sim \gamma \log n$ and $\widetilde{\Phi}(n) \sim c \log^{1/\gamma} n$; as at the end of Section 2.4, we do not assume that $\mathtt{m} = \mathbb{E}\widetilde{S}_1$ takes the value 1, and we write $\tau^2 = \operatorname{Var} \widetilde{S}_1$. Theorem 6.2 entails a gaussian limit for $(K_n - \mu_n)/\sigma_n$, with

$$\mu_n \sim \log n \int_0^1 \Phi(n^{1-v})\, \mathrm{d}v \ \sim \ \frac{c \log^{1+1/\gamma} n}{\mathtt{m}(1+1/\gamma)}$$

and

$$\sigma_n^2 \ \sim \ \Phi^2(n) \log n\, \sigma^2 \operatorname{Var}\left\{\int_0^1 \gamma^{-1}(1-v)^{1/\gamma - 1} W(v) \mathrm{d}v\right\} \ \sim \ \frac{c^2\tau^2 \log^{1+2/\gamma} n}{\mathtt{m}^3(1+2/\gamma)},$$

where, as before, $\Phi(n) = \mathtt{m}^{-1}\widetilde{\Phi}(n)$ and $\sigma^2 = \mathtt{m}^{-1}\tau^2$. Note also that $\mu_n \sim \Psi(n) \sim \mathtt{m}^{-1}\widetilde{\Psi}(n)$ and that $\sigma_n^2 \sim \sigma^2 \Psi_2(n) \sim \tau^2 \mathtt{m}^{-3} \widetilde{\Psi}_2(n)$, as is to be expected.

For the classical gamma subordinator [2, p. 73], scaling so that $\mathbb{E}S_1 = 1$, we have $\nu_0(\mathrm{d}x) = \theta e^{-\theta x} \mathrm{d}x/x$, $\Phi_0(n) = \theta \log(1 + n/\theta)$, and $\sigma^2 = 1/\theta$. Hence the CLT in [11] agrees with Theorem 6.2. Note that one parameter $\theta > 0$ is enough, since, for the Lévy measure $a\nu_0$, the distribution of K_n does not depend on the scale parameter a.

In the case $\gamma = 1$, Theorem 6.2 covers a somewhat larger family of gamma-like subordinators than that considered in [11]. The extension is that the condition of exponential decay for $N_0(x)$ as $x \to \infty$ required in [11] is replaced now by a weaker condition $\sigma^2 < \infty$. The constraints on the behaviour of $N(x)$ at $x \to 0$ are also slightly weaker here.

6.2 Fast growth

We now turn to the setting in which $L(n) \to \infty$ but $L(n)/\log n \to 0$; hence Φ grows faster than any power of the logarithm. In order to apply the previous theorems, we need to suppose either that (4.36) is in force, albeit with $L(n) = o(\log n)$, or that (4.47) holds. The analogue of $\mathcal{K}_n^{(1)}$ is now $\mathcal{K}_n^{(2)}$, defined by

$$\mathcal{K}_n^{(2)}(u) \;:=\; \left(\Phi(n)\sqrt{L(n)}\right)^{-1}\left\{\mathcal{K}_n(uL(n)) - L(n)\int_0^{(u\wedge l_n)} \Phi(ne^{-vL(n)})\,\mathrm{d}v\right\},$$

where $l_n := \log n/L(n)$. Here, we approximate the distribution of $\mathcal{K}_n^{(2)}$ by that of $Y^{(2)}$, where

$$Y^{(2)}(u) \;:=\; \sigma\int_0^u e^{-v}\,W(v)\,\mathrm{d}v.$$

Theorem 6.3 *If Assumptions* A1–A2 *hold, and* $L(n)/\log n \to 0$, *with either* (4.36) *or* (4.47) *satisfied, then*

$$d_{BW}(\mathcal{L}(\mathcal{K}_n^{(2)}), \mathcal{L}(Y^{(2)})) \;\to\; 0 \quad \textit{as } n \to \infty,$$

where d_{BW} *is as before.*

Proof. If (4.36) is satisfied, we argue as in the proof of Theorem 6.1 to show that

$$\sup_{u\geq 0}|\mathcal{K}_n^{(2)}(u) - \widetilde{Y}_n^{(2)}(u)| \;\to_p\; 0 \quad \text{as } n \to \infty, \tag{6.74}$$

where

$$\widetilde{Y}_n^{(2)}(u) \;:=\; \sigma\int_0^{(u\wedge l_n)} h_n^{(2)}(v)W_{L(n)}(v)\,\mathrm{d}v,$$

and

$$h_n^{(2)}(u) \;:=\; \frac{\Phi(ne^{-uL(n)})L(n)}{\Phi(n)L(ne^{-uL(n)})}.$$

If (4.47) is satisfied, (6.74) is still true, using Theorem 4.6 in place of Theorem 4.4 in the proof. Once again, $h_n^{(2)}(u) \geq 0$ for all u, and

$$\int_0^\infty h_n^{(2)}(u)\,\mathrm{d}u \;=\; 1.$$

The next step is to approximate $\widetilde{Y}_n^{(2)}$ by $Y_n^{(2)}$, where

$$Y_n^{(2)}(u) \;:=\; \sigma\int_0^u e^{-v}W_{L(n)}(v)\,\mathrm{d}v.$$

Here, it is immediate that

$$\mathbb{E}\left\{\sup_{u\geq 0}\sigma^{-1}|\widetilde{Y}_n^{(2)}(u) - Y_n^{(2)}(u)|\right\} \;\leq\; \int_0^{l'_n} |h_n^{(2)}(v) - e^{-v}|\sqrt{v}\,\mathrm{d}v + \int_{l'_n}^\infty \{h_n^{(2)}(v) + e^{-v}\}\sqrt{v}\,\mathrm{d}v \tag{6.75}$$

for any $l'_n \le l_n$; we take $l'_n = \min\{l_n^{1/2}, \frac{1}{2(\varkappa\vee 1)} l_n\}$. Now, from (3.26), for $0 \le v \le l'_n$, we have

$$\left| \frac{L(n)}{L(ne^{-uL(n)})} - 1 \right| \;\le\; \lambda_1 l_n^{-1/2},$$

for some $\lambda_1 < \infty$, and hence, by Corollary 3.3, that

$$\sqrt{v}\,\bigl|h_n^{(2)}(v) - \Phi^*(n,v)\bigr| \;\le\; \lambda_2 \sqrt{v} e^{-v/2} l_n^{-1/2},$$

where

$$\Phi^*(n,v) \;:=\; \frac{\Phi(ne^{-vL(n)})}{\Phi(n)}.$$

Then Lemma 3.2 gives

$$\sqrt{v}\,\bigl|\Phi^*(n,v) - e^{-v}\bigr| \;\le\; \lambda_3\{\exp(\varkappa v^2/l_n) - 1\}\sqrt{v} e^{-v} \;\le\; \lambda_4 l_n^{-1} v^{5/2} e^{-v},$$

for different constants λ_3, λ_4, again in $0 \le v \le l'_n$. Hence it follows that

$$\lim_{n\to\infty} \int_0^{l'_n} |h_n^{(2)}(v) - e^{-v}| \sqrt{v}\,\mathrm{d}v \;=\; 0. \tag{6.76}$$

It is also immediate that $\lim_{n\to\infty} \int_{l'_n}^{\infty} e^{-v}\sqrt{v}\,\mathrm{d}v = 0$. Hence, to show that the right hand side of (6.75) is small in the limit, it remains only to consider

$$\int_{l'_n}^{\infty} h_n^{(2)}(v)\sqrt{v}\,\mathrm{d}v \;=\; \sqrt{l'_n}\,\Phi^*(n,l'_n) + \int_{l'_n}^{\infty} \tfrac{1}{2} v^{-1/2} \Phi^*(n,v)\,\mathrm{d}v. \tag{6.77}$$

Here, the first term tends to zero as $n \to \infty$ by Corollary 3.3, as does

$$\int_{l'_n}^{\alpha_n} v^{-1/2}\Phi^*(n,v)\,\mathrm{d}v \;\le\; \int_{l'_n}^{\alpha_n} v^{-1/2} e^{-v/2}\,\mathrm{d}v,$$

where $\alpha_n := l_n/\{2(\varkappa \vee 1)\}$. Then, splitting the remaining integral at $2l_n$, we have

$$\int_{\alpha_n}^{\infty} v^{-1/2}\Phi^*(n,v)\,\mathrm{d}v \;\le\; 2l_n \exp\{-\alpha_n/2\} + \frac{1}{n\Phi(n)L(n)}\,,$$

the final term following from (2.6). Combining these bounds, we have now also shown that $\lim_{n\to\infty} \int_{l'_n}^{\infty} h_n^{(2)}(v)\sqrt{v}\,\mathrm{d}v = 0$; hence, from (6.74) and (6.75), it follows that

$$d^0_{BW}(\mathcal{L}(\mathcal{K}_n^{(2)}), \mathcal{L}(Y_n^{(2)})) \;\to\; 0 \quad \text{as } n \to \infty.$$

Finally, if $H\colon D_1[0,\infty[\to D_0[0,\infty[$ is defined by $H(w)(u) := \int_0^u e^{-v} w(v)\,\mathrm{d}v$ and f is in $\mathcal{W}^0$, then $(1+e^{-1})^{-1} f \circ H \in \mathcal{W}^1$, from which $d^0_{BW}(\mathcal{L}(Y_n^{(2)}), \mathcal{L}(Y^{(2)})) \to 0$ as $n \to \infty$ follows immediately, and the theorem is proved. □

6.3 Slow growth

If Φ grows very slowly to infinity, with $L(n)/\log n \to \infty$, the arguments culminating in Theorem 5.5 show that the key quantity describing the process $\mathcal{K}_n$ is the family of integrals

$$\int_{(\tau_n - t)_+}^{\tau_n} \Phi(e^v)\,dv, \quad t \geq 0.$$

Here, the randomness enters only through the hitting time τ_n, which is asymptotically normally distributed, as recorded in (6.73). The process thus has a quite different qualitative behaviour to that of the previous cases.

Since τ_n takes values fairly close to $\log n$, it makes sense to describe the random behaviour of $\mathcal{K}_n(t)$ by first subtracting $\int_{(\log n - t)_+}^{\log n} \Phi(e^v)\,dv$, and then dividing by $\Phi(n)\sqrt{\log n}$. This leads us to define the process $\mathcal{K}_n^{(3)}$ for $t \geq 0$ by

$$\mathcal{K}_n^{(3)}(t) \;:=\; \Big(\Phi(n)\sqrt{\log n}\Big)^{-1} \left\{ \mathcal{K}_n(t) - \int_{(\log n - t)_+}^{\log n} \Phi(e^v)\,dv \right\}.$$

Then, defining $G_n \colon \mathbb{R} \to D_0[0,\infty[$ by

$$G_n[u](t) \;:=\; \sigma u - \Big(\Phi(n)\sqrt{\log n}\Big)^{-1} \int_{(\log n - t)_+}^{(\log n - t + \sigma u\sqrt{\log n})_+} \Phi(e^v)\,dv, \quad t \geq 0, \qquad (6.78)$$

for each $u \in \mathbb{R}$, we define our approximating process to be $Y_n^{(3)} = G_n[U]$, where U is a standard normal random variable.

Theorem 6.4 *Suppose that Assumptions* A1–A2 *and* (5.52) *hold. Then it follows that*

$$d_{BW}^0(\mathcal{L}(\mathcal{K}_n^{(3)}), \mathcal{L}(Y_n^{(3)})) \;\to\; 0 \quad \text{as } n \to \infty.$$

Proof. Once again, we combine Lemmas 2.2 and 2.6 and Corollary 3.4, this time with Theorem 5.5, showing that

$$d_{BW}^0(\mathcal{L}(\mathcal{K}_n^{(3)}), \mathcal{L}(\widetilde{Y}_n^{(3)})) \;\to\; 0 \quad \text{as } n \to \infty,$$

where

$$\begin{aligned}
\widetilde{Y}_n^{(3)}(t) \;&:=\; \Big(\Phi(n)\sqrt{\log n}\Big)^{-1} \left\{ \int_{(\tau_n - t)_+}^{\tau_n} \Phi(e^v)\,dv - \int_{(\log n - t)_+}^{\log n} \Phi(e^v)\,dv \right\} \\
&=\; \Big(\Phi(n)\sqrt{\log n}\Big)^{-1} \left\{ \int_{\log n}^{\tau_n} \Phi(e^v)\,dv - \int_{(\log n - t)_+}^{(\tau_n - t)_+} \Phi(e^v)\,dv \right\}.
\end{aligned}$$

Now, from (5.69), we have

$$\mathbf{1}\{B_0(n)\} \Big(\Phi(n)\sqrt{\log n}\Big)^{-1} \left| \int_{\log n}^{\tau_n} \Phi(e^v)\,dv - \int_{\log n}^{\tau_n} \Phi(n)\,dv \right| \;\leq\; |(2^{1/c_2^*(\sqrt{n})} - 1)U_n|,$$

and this tends to zero as $n \to \infty$ because of (6.73) together with $\lim_{n\to\infty} c_2^*(n) = \infty$. On the other hand, $\lim_{n\to\infty} \mathbb{P}[B_0(n)] = 0$, by Lemma 4.1. Hence $d^0_{BW}(\mathcal{L}(\widetilde{Y}_n^{(3)}), \mathcal{L}(G_n[U_n])) \to 0$ as $n \to \infty$.

To complete the proof, we just have to show that the distributions of $G_n[U_n]$ and $G_n[U]$ are close. To do so, we first define $\widetilde{G}_n \colon \mathbb{R} \to D_0[0,\infty[$ by $\widetilde{G}_n[u] = G_n[u \wedge \sigma^{-1}\sqrt{\log n}]$. Then, from the definition of G_n and from (5.69), it is immediate that

$$\begin{aligned}\sup_{v\ge 0} |\widetilde{G}_n[u'](v) - \widetilde{G}_n[u](v)| &\le \sigma|u'-u| + \left(\Phi(n)\sqrt{\log n}\right)^{-1} \Phi(n^2)\sigma\sqrt{\log n}|u'-u| \\ &\le 3\sigma|u'-u|;\end{aligned}$$

hence $\lim_{n\to\infty} \sup_{f\in\mathcal{W}^0} |\mathbb{E}f(\widetilde{G}_n[U_n]) - \mathbb{E}f(\widetilde{G}_n[U])| = 0$, in view of (6.73). Finally,

$$\begin{aligned}&\sup_{f\in\mathcal{W}^0} \{|\mathbb{E}f(G_n[U_n]) - \mathbb{E}f(\widetilde{G}_n[U_n])| + |\mathbb{E}f(G_n[U]) - \mathbb{E}f(\widetilde{G}_n[U])|\} \\ &\quad\le\ 2\mathbb{P}[U_n > \sigma^{-1}\sqrt{\log n}] + 2\mathbb{P}[U > \sigma^{-1}\sqrt{\log n}] \ \to\ 0,\end{aligned}$$

and the theorem follows. □

The process $Y_n^{(3)}$ starts close to zero, and, as indicated by Lemmas 5.1 and 5.2, remains close to zero until $1 - t/\log n$ becomes small. It reaches its final value σU at time $\log n + \sigma U\sqrt{\log n}$ if $U \ge 0$, and at time $\log n$ if $U < 0$.

Its behaviour can also be understood in terms of the overlapping representation provided under the condition (4.46), when $c_2(n)$ is allowed to tend to infinity, but not too fast. Here, the approximation to the random fluctuations is expressed in terms of the process

$$\left(\Phi(n)\sqrt{\log n}\right)^{-1} \int_0^{T\wedge\log n} \Phi(ne^{-t})\frac{Z_t}{L(ne^{-t})}\,dt,$$

which at first sight looks very different. Here, however, as already observed at the start of Section 5,

$$\left(\Phi(n)\sqrt{\log n}\right)^{-1} \int_0^{(u\log n)\wedge\log n} \Phi(ne^{-t})\frac{Z_t}{L(ne^{-t})}\,dt$$

is of small order whenever u is bounded away from 1, and even for choices of $u = u(n) \to 1$ such that $(1-u(n))c_2(n^{1-u(n)}) \to \infty$. On the other hand, for u closer to 1, the remaining contribution is approximately

$$\begin{aligned}&\left(\Phi(n)\sqrt{\log n}\right)^{-1} Z_{\log n} \int_{u(n)\log n}^{u\log n} \frac{\Phi(ne^{-t})}{L(ne^{-t})}\,dt \\ &\quad=\ \left(\Phi(n)\sqrt{\log n}\right)^{-1} Z_{\log n}\{\Phi(n^{1-u(n)}) - \Phi(n^{1-u})\} \\ &\quad\approx\ \left\{1 - \frac{\Phi(n^{1-u})}{\Phi(n)}\right\}\sigma W_{\log n}(1),\end{aligned}$$

whose randomness is determined only by the value of $W_{\log n}(1) \sim -(\tau_n - \log n)/\sigma\sqrt{\log n}$. To match this with the corresponding formula for $Y_n^{(3)}(t)$, note that, under (4.46), the second term in $G_n[U_n]$,

$$\left(\Phi(n)\sqrt{\log n}\right)^{-1}\int_{(\log n-t)_+}^{(\log n-t+\sigma U_n\sqrt{\log n})_+}\Phi(e^v)\,\mathrm{d}v\,,$$

is small for $\log n-t=O(\sqrt{\log n})$, and that, for larger values of $\log n-t=(1-u)\log n$, one can replace $\Phi(e^v)$ by $\Phi(n^{1-u})$ in the integral.

Remark. Setting formally $\Phi(n)=\text{const}$ in the above formulas suggests that that $K_n\sim\tau_n$ in the case of bounded ν_0. The latter is indeed true and, moreover, $|K_n-\tau_n|$ remains bounded with all moments as n grows; the reason for this behaviour in the compound Poisson case is just that essentially all gaps within $\mathcal{R}\cap[0,\log n]$ are hit by the atoms of Y_n, hence K_n is close to the number of renewals on $[0,\log n]$.

References

[1] R. Arratia, A.D. Barbour and S. Tavaré. *Logarithmic Combinatorial Structures: A Probabilistic Approach*, European Math. Soc. Monographs in Math., v. 1, 2003.

[2] J. Bertoin, *Lévy Processes*, Cambridge University Press, 1996.

[3] J. Bertoin, *Subordinators: Examples and Applications*, Springer Lecture Notes in Math. vol. 1727, 1996

[4] N.H. Bingham, C.M. Goldie and J.L. Teugels. *Regular Variation*, Cambridge University Press, 1987.

[5] R.M. Dudley. *Probabilities and metrics*, Lecture Notes Series **45**, Aarhus Universitet, 1976.

[6] W. Feller, *An Introduction to Probability Theory and its Applications*, volume II, Wiley, 2nd edition, 1971.

[7] A.V. Gnedin. The Bernoulli sieve, *Bernoulli* 10: 79–96, 2004.

[8] A.V. Gnedin and J. Pitman. Regenerative composition structures, *Ann. Probab.* 33: 445–479, 2005.

[9] A.V. Gnedin and J. Pitman. Regenerative partition structures, *Elec. J. Combin.* 11(2), paper #R12, 2005.

[10] A.V. Gnedin, J. Pitman and M. Yor. Asymptotic laws for compositions derived from transformed subordinators, *Ann. Probab.* (to appear), http:// arxiv.org/abs/math.PR/0403438, 2004.

[11] A.V. Gnedin, J. Pitman and M. Yor. Asymptotic laws for regenerative compositions: gamma subordinators and the like, http:// arxiv.org/abs/math.PR/0405440, 2004.

[12] G. Last and A. Brandt. *Marked Point Processes on the Real Line: The Dynamic Approach*, Springer, NY, 1995.

[13] D.W. Müller. Verteilungs-Invarianzprinzipien für das starke Gesetz der großen Zahl, *Z. Wahrscheinlichkeitstheorie verw. Geb.* 10: 173–192, 1968.

[14] J. Pitman, *Combinatorial Stochastic Processes*, Springer Lecture Notes Math. (to appear), 2002.

[15] M. Winkel, Electronic foreign exchange markets and level passage events of multivariate subordinators, *J. Appl. Prob.* (to appear)